

COMPLETE ALGEBRAIC VECTOR FIELDS ON AFFINE SURFACES

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ABSTRACT. Let $\text{AAut}_{\text{hol}}(X)$ be the subgroup of the group $\text{Aut}_{\text{hol}}(X)$ of holomorphic automorphisms of a normal affine algebraic surface X generated by elements of flows associated with complete algebraic vector fields. Our main result is a classification of all normal affine algebraic surfaces X quasi-homogeneous under $\text{AAut}_{\text{hol}}(X)$ in terms of the dual graphs of the boundaries $\bar{X} \setminus X$ of their SNC-completions \bar{X} .

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1. INTRODUCTION

1.1. General Motivation. In the last decades affine algebraic varieties and Stein manifolds with big (infinite-dimensional) automorphism groups have been studied intensively. Several notions expressing the fact that the automorphisms group of a manifold is big have been proposed. Among the most important of them are (algebraic) density property and holomorphic flexibility with the former implying the latter. Both density property and holomorphic flexibility show that the manifold in question is an Oka-Forstnerič manifold. This important notion has also recently merged from the intensive studies around the homotopy principle which goes back to the 1930's and has had an enormous impact on the development of Complex Analysis with a constantly growing number of applications (for definitions and more information we refer the reader to [12]).

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In spite of the large number of examples of such highly symmetric objects their classification and the exact relations between all mentioned properties remain unclear even in dimension 2. In particular, we do not know description of Stein surfaces X on which the group of holomorphic automorphisms $\text{Aut}_{\text{hol}}(X)$ acts transitively. Meanwhile such transitivity is an automatic consequence of flexibility and it has to be studied.

In algebraic case the similar question was “almost” resolved in the papers of Gizatullin and Danilov [13], [15] and we need the following definition to formulate their result.

Definition 1.1. We call a normal Stein (resp. affine algebraic) surface X quasi-homogeneous with respect to a subgroup G of the group of its holomorphic (resp. algebraic) automorphisms if the natural action of G has an open orbit in X whose complement is at most finite. A normal algebraic surface is called quasi-homogeneous (without any reference to a group) if it is quasi-homogeneous with respect to the group $\text{Aut}_{\text{alg}}(X)$ of algebraic automorphisms¹.

With the exception of the two-dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathbb{C} \times \mathbb{C}^*$ every normal open quasi-homogeneous surface has an SNC-completion \bar{X} such that the dual graph Γ of its boundary $\bar{X} \setminus X$ is a linear rational graph [13], [15] which can be always chosen in the following standard form (so-called standard zigzag)

$$\begin{array}{ccccccc} C_0 & C_1 & C_2 & & & & C_n \\ \circ & \circ & \circ & \cdots & \circ & & \circ \\ 0 & 0 & w_2 & & & & w_n \end{array}$$

where $n \geq 0$ and $w_i \leq -2$ for $i = 2, \dots, n$. A surface that admits such a completion will be called below a Gizatullin surface². Let $\text{SAut}(X)$ be the subgroup of $\text{Aut}_{\text{alg}}(X)$ generated by element of G_a -actions. Then this subgroup possesses already an open orbit in X whose complement is at most finite. Recall that G_a -action can be viewed as the phase flow of a complete algebraic vector field on X that is locally nilpotent.

Recall that a holomorphic vector field ν on a complex space X is called complete if the solution of the ODE

$$\frac{d}{dt}\varphi(x, t) = \nu(\varphi(x, t)) \quad \varphi(x, 0) = x$$

is defined for all complex times $t \in \mathbb{C}$ and all initial values $x \in X$. The induced maps $\Phi_t : X \rightarrow X$ given by $\Phi_t(x) = \varphi(x, t)$ yield the phase flow of ν which is nothing but a one-parameter subgroup in the group $\text{Aut}_{\text{hol}}(X)$ of holomorphic automorphism with parameter $t \in \mathbb{C}_+$ (so-called holomorphic \mathbb{C}_+ -action).

¹ When the complement to the open orbit is empty one has transitivity. However there are examples of smooth quasi-homogeneous surfaces for which the complements of the open orbits are not empty. In the case of surfaces over algebraically closed field of positive characteristic they appeared already in the paper of Gizatullin and Danilov [14] who also knew but did not publish such examples for characteristic zero. In a published form examples of complex quasi-homogeneous surfaces with non-empty complements can be found in a recent paper of Kovalenko [21].

² The class of Gizatullin surfaces coincides also with the class of surfaces with a trivial Makar-Limanov invariant, i.e. $\text{ML}(X) = \mathbb{C}$ where $\text{ML}(X)$ is the subring of the ring of regular functions on X that consists of all functions invariant under any (algebraic) G_a -action on X .

One could wish to extend the quasi-homogeneity results to the analytic situation replacing locally nilpotent vector fields by complete holomorphic vector fields and $\text{SAut}(X)$ by its holomorphic analogue, but unfortunately classification of complete holomorphic vector fields on Stein surfaces with sufficiently big automorphism groups (even on \mathbb{C}^2) seems still far out of reach. However complete algebraic vector fields have been classified on $\mathbb{C}^* \times \mathbb{C}^*$ (actually on $(\mathbb{C}^*)^n$) by Andersén using Nevanlinna theory [1] and on \mathbb{C}^2 by Brunella [5] using foliation theory. It is worth mentioning that when X is an affine algebraic variety and ν is a complete algebraic vector field its phase flow may well be non-algebraic (when the phase flow is algebraic then either ν is locally nilpotent and we have a G_a -action or ν is semi-simple and we have a G_m -action).

Definition 1.2. A holomorphic automorphism α of an algebraic variety X will be called *algebraically generated* if α coincides with an element Φ_t of the phase flow of a complete algebraic vector field on X as before. The subgroup of the holomorphic automorphism group generated by such algebraically generated automorphisms will be denoted by $\text{AAut}_{\text{hol}}(X)$ and a normal affine algebraic surface X quasi-homogeneous with respect to $\text{AAut}_{\text{hol}}(X)$ will be called *generalized Gizatullin surface*. If normal affine algebraic surface Y admits two complete non-proportional algebraic vector fields ν_1 and ν_2 (i.e. $f_1\nu_1 \neq f_2\nu_2$ for any pair nonzero regular functions f_1 and f_2 on Y) then there is an open orbit of the natural $\text{AAut}_{\text{hol}}(Y)$ -action in Y . In what follows we call such Y a *surface with an open orbit* (without mentioning $\text{AAut}_{\text{hol}}(Y)$). Of course, every *generalized Gizatullin surface* is a surface with an open orbit.

In this paper we deal with the following first step of our program in dimension 2.

1.2. Complete algebraic vector fields and quasi-homogeneity. Let us formulate two main results of this paper.

Theorem 1.3. *Let X be a normal affine algebraic surface which admits a nonzero complete algebraic vector field. Then either:*

- (1) *all complete algebraic fields share the same rational first integral (i.e. there is a rational map $f : X \dashrightarrow B$ such that all complete algebraic vector fields on X are tangent to the fibers of f), or*
- (2) *X is a rational surface with an open orbit and, furthermore, for every complete algebraic vector field ν on X there is a regular function $f : X \rightarrow \mathbb{C}$ (depending on ν) with general fibers isomorphic to \mathbb{C} or \mathbb{C}^* such that the flow of ν sends fibers of f to fibers of f .*

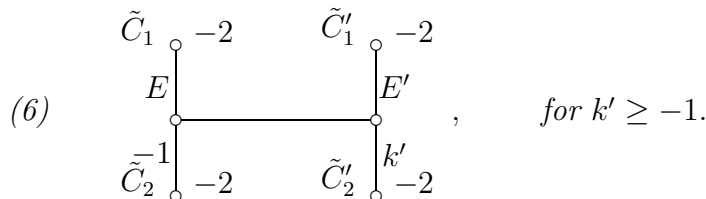
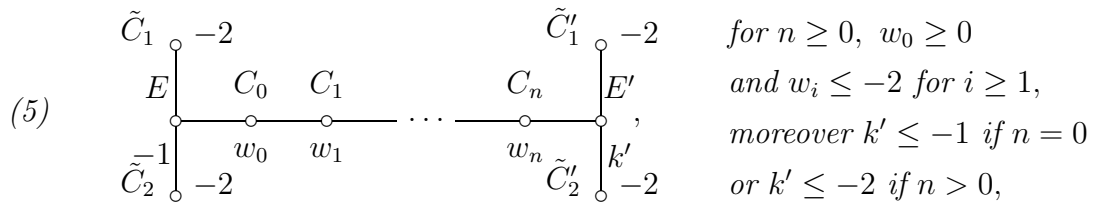
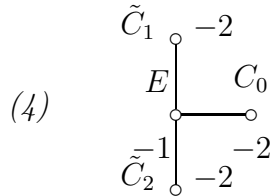
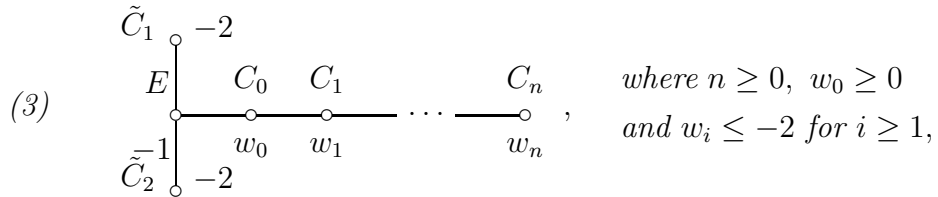
The fact that the flow sends fibers to fibers can be reformulated as follows: there is a complete vector field ν_0 on \mathbb{C} such that f is \mathbb{C}_+ -equivariant with respect to the phase flows of ν (resp. ν_0) acting on X (resp. \mathbb{C}). When ν_0 is trivial then f is again a rational first integral of ν (which is regular in this case)

In the special case of $X = \mathbb{C}^2$ Theorem 1.3 was proven by Brunella [5] whose paper has ramifications beyond our result. In particular, after finishing the first version of the manuscript we were informed about the paper of Guillot and Rebello [16] which is also heavily based on the technique of McQuillan and Brunella. They proved an analogue of Theorem 1.3 for so called semicomplete meromorphic vector fields on complex surfaces in which the assumption on the isomorphism type of a general fiber of f in Theorem 1.3

(2) is replaced by the following: the completion of such a fiber is either a rational curve or an elliptic one. Complete algebraic fields on affine algebraic surfaces are, of course, semicomplete, but completeness enables us to give this stronger statement which in turn leads to our second main result.

Theorem 1.4. *A normal complex affine algebraic surface X is generalized Gizatullin if and only if it admits an SNC-completion \bar{X} for which the boundary $\bar{X} \setminus X$ is connected, consists of rational curves, and has a dual graph that belongs to one of the following types*

- (1) *a standard zigzag or a linear chain of three 0-vertices (i.e. Gizatullin surfaces and $\mathbb{C} \times \mathbb{C}^*$),*
- (2) *circular graph with the following possibilities for weights*
 - (2a) $((0, 0, w_1, \dots, w_n))$ where $n \geq 0$ and $w_i \leq -2$,
 - (2b) $((0, 0, w))$ with $-1 \leq w \leq 0$ or $((0, 0, 0, w))$ with $w \leq 0$,
 - (2c) $((0, 0, -1, -1))$;



It is worth mentioning that the first examples of generalized Gizatullin surfaces with discrete algebraic automorphism group $\text{Aut}_{\text{alg}}(X)$ were found by the first and second author in [19], [20] and they can be presented as hypersurfaces $\{xp(x) + yq(y) + xyz = 1\} \subset \mathbb{C}_{x,y,z}^3$ where polynomials $1 - xp(x)$ and $1 - yq(y)$ have simple roots only (none of them admits nontrivial algebraic G_a or G_m -actions). Similar to the case of torus for each such a surface X an SNC-completion \bar{X} can be chosen as a cycle.

Furthermore, in the framework of our general program Theorem 1.4 provides us with a list³ of affine algebraic surfaces on which one can look for holomorphic flexibility, algebraic density property, or even classification of all complete algebraic vector fields. A case of particular surfaces has been already worked out by the third author in [23].

Let us briefly discuss the technique we use. Since any algebraic vector field ν on a normal affine algebraic surface X induces a foliation \mathcal{F} on an SNC-completion \bar{X} of X we use the fundamental results of foliation theory for surfaces due to Suzuki [33], [34], McQuillan [26], [25], and Brunella [5]. In the last paper Brunella classified all complete algebraic vector fields on the complex plane \mathbb{C}^2 . Apart from the case when a complete algebraic vector field ν on X has a rational first integral the approach of Brunella (which is applicable not only to the complex plane but to any generalized Gizatullin surface) can be described as follows.

After additional blowing-ups $\hat{X} \rightarrow \bar{X}$ one can suppose that the induced foliation $\hat{\mathcal{F}}$ on \hat{X} has reduced singularities only and \hat{X} is smooth. Then it admits so-called McQuillan contraction $\theta : \hat{X} \rightarrow \hat{X}'$ such that the foliation $\hat{\mathcal{F}}'$ generated on \hat{X}' has a nef canonical bundle $K_{\hat{\mathcal{F}}'}$. In the case when there is no rational first integral the crucial result of McQuillan implies that for a complete ν the Kodaira dimension $\text{kod}(\hat{\mathcal{F}}')$ of the foliation is either 1 or 0. For the case of $\text{kod}(\hat{\mathcal{F}}') = 1$ the results of McQuillan and Brunella imply that $\hat{\mathcal{F}}'$ is a Riccati foliation which yields a morphism $f : X \rightarrow B$ with general fibers either \mathbb{C} or \mathbb{C}^* such that the phase flow of ν transforms each fiber of f into a fiber. In the remaining cases, namely $\text{kod}(\hat{\mathcal{F}}') = 0$ or existence of rational first integral (this case is obvious) we also deduce the existence of such a morphism f . Moreover we show that unless all complete fields on X share the same first integral f can be chosen as a regular function. In the presence of such an f Theorem 1.3 becomes transparent as well as the proof of generalized quasi-homogeneity since one can use now the technique of \mathbb{P}^1 -fibrations on complete surfaces.

The paper is organized as follows. In the Section 2 we remind some facts from [8] (see also [6], [7]) about weighted dual graphs of algebraic curves contained in smooth surfaces. In Section 3 we present basic information on the foliation theory which can be found mostly in Brunella's surveys [3], [4]. In Section 4 we explain why Brunella's results about foliations on \mathbb{C}^2 with Kodaira dimension 1 can be transferred to foliations on smooth semi-affine surfaces (which are desingularizations of normal affine surfaces). Section 5 contains a proof that every foliation of Kodaira dimension 0 that is induced

³It should be emphasized that in general surfaces with the same graph in Theorem 1.4 are not necessarily homeomorphic. Even in the case of the same topology such surfaces may admit families with non-isomorphic members (we do not know if a similar fact holds in the analytic setting). Furthermore, homogeneity of such surfaces with respect to the AAut_{hol} -action does not guarantee algebraic density property. For instance, the hypersurface $\{xp(x) + yq(y) + xyz = 1\} \subset \mathbb{C}_{x,y,z}^3$ mentioned before does not have it, though it has another positive feature - the algebraic volume density property [20].

by a complete algebraic vector field on a smooth semi-affine surface admits a morphism $f : X \rightarrow B$ with general fibers either \mathbb{C} or \mathbb{C}^* such that the phase flow of ν transforms each fiber of f into a fiber. In Section 6 we present some basic results on \mathbb{P}^1 -fibrations on \hat{X} extending \mathbb{C} - or \mathbb{C}^* -fibrations on X . Sections 7 and 8 are devoted to geometrical description of rational first integrals of complete algebraic vector fields on smooth semi-affine surfaces (whenever such integrals exist) which together with the results from the sections before allows us to complete the proof of Theorem 1.3. In Section 9 we show there is no graph different from those described in (1)-(6) that can serve as the dual graph of a boundary of a surfaces quasi-homogeneous under $\text{AAut}_{\text{hol}}(X)$. In Section 10 we establish that any surface with a boundary graph as in (1)-(6) is indeed quasi-homogeneous under $\text{AAut}_{\text{hol}}(X)$ which concludes the proof of Theorem 1.4. In the last Section 11 we show that some of surfaces in Theorem 1.4 are in fact $\text{AAut}_{\text{hol}}(X)$ -homogeneous. In particular, this class includes all surfaces of type (2) and all known Gizatullin surfaces that are not homogeneous with respect to the natural $\text{Aut}(X)$ -action described in [21].

2. PRELIMINARIES ABOUT DUAL GRAPHS

In this section we discuss some facts about weighted dual graphs of algebraic curves contained in smooth complete surfaces (e.g., see [8]).

Definition 2.1. Let D be a closed curve contained in a smooth complete algebraic surface \bar{X} such that all of its singularities are simple nodes (i.e. locally each of these singularities is a transversal intersection of two smooth analytic branches). Then we can assign the so-called weighted dual graph Γ such that

- (1) its vertices are in bijective correspondence with the irreducible components of D ;
- (2) every singularity of D corresponds to an edge that joins vertices corresponding to irreducible components C_1 and C_2 that contain this singularity (if $C_1 = C_2$ then this edge is a loop);
- (3) each vertex is equipped with a weight that is the integer equal to the selfintersection number C^2 of the corresponding component C in \bar{X} .
- (4) We also say that D is of simple normal crossing type if all components are smooth. This implies that the dual graph Γ does not have loops.

Convention 2.2. From now on we identify the vertices of Γ and the corresponding components of D and denote them by the same letters. Furthermore, we may treat a curve contained in D as a subgraph of Γ and vice versa.

Recall that the valency of a vertex $C \in \Gamma$ is the number of edges adjacent to this vertex (with each loop counting twice) and the vertices joined with C by edges are called the neighbors of C . The vertex is an end vertex (resp. linear vertex, resp. branch point) if its valency is 1 (resp. 2; resp. at least 3). The graph is called linear or a chain if it does not contain branch points but contains an end vertex. We use notation as $C_1 + C_2 + \dots + C_n$ to denote such a chain with n vertices in the natural order. If the weight of each C_i is w_i we shall also use notation $[[w_1, w_2, \dots, w_n]]$ for this chain. A graph without branch points and end vertices is called circular. In this case we write $((w_1, w_2, \dots, w_n))$ for the weights of this graph in a counterclockwise order.

For any subgraph Γ_0 of Γ notation $\Gamma \ominus \Gamma_0$ will be used to denote the graph obtained from Γ by removing each vertex $C \in \Gamma_0$ and its adjacent edges.

If C is a rational irreducible component of D with selfintersection k we call C a k -vertex. If C is a (-1) -vertex with valency at most 2 in Γ then it can be contracted and the image D' of D is still a curve with nodes as singularities (unless C with a loop is a connected component of Γ - the case which we do not consider). The graph Γ' of D' in the smooth resulting surface can be obtained from $\Gamma \ominus C$ by joining the distinct neighbors of C by an edge and increasing their weights by 1 (we call such replacement of Γ by Γ' a blowing down).

A graph Γ is contractible if it can be reduced to an empty graph by a sequence of blowing down (i.e D can be blown down to a smooth point of a resulting surface). We call Γ minimal if it does not contain (-1) -vertices different from branch points.

Let $z \in D$, $\sigma : \hat{X} \rightarrow \bar{X}$ be the monoidal transformation of \bar{X} at z , and $D'' = \sigma^{-1}(D)$. Then the form of the dual graph Γ'' of D'' in \hat{X} depends on whether z is (a) a smooth point of D (and in particular z is a smooth point of an irreducible component C of D) or (b) a double point of D , i.e. $z \in C_1 \cap C_2$ where C_1 and C_2 are irreducible components of D . In case (a) Γ'' is obtained from Γ by creating a new vertex of weight -1 , joining it with C , and reducing the weight of C by 1. This change of Γ will be called an outer blowing up. In case (b) Γ'' is obtained from Γ by removing the edge between C_1 and C_2 , reducing their weights by 1, and joining them by edges with a new vertex of weight -1 . Such a change of Γ will be called an inner blowing up.

If a graph Γ_2 can be obtained from a graph Γ_1 by a sequence of blowing up and blowing down then we call this procedure a reconstruction of Γ_1 into Γ_2 . Let us give some example of reconstructions. If C is of nonnegative weight in a $\Gamma \neq C$ then making inner blowing up at an edge of C one can make its weight 0. If C is a linear 0-vertex with neighbors of weight w_1 and w_2 then making an inner blowing up at an edge of C and contracting C we get a reconstruction $[[w_1, 0, w_2]] \rightarrow [[w_1 - 1, 0, w_2 + 1]]$. Similarly if C is end 0-vertex with a neighbor of weight w one can get $[[0, w]] \rightarrow [[0, w + 1]]$ or $[[0, w]] \rightarrow [[0, w - 1]]$. The last three reconstructions around a 0-vertex are called elementary transformations. The next straightforward fact will be useful.

Proposition 2.3. (see also [8, Section 2]). (1) *Let $C_1 + C_2 + C_3$ be a chain with weights $w_1, 0, w_2$. Then there exists elementary transformations such that C_1 and C_3 are not blown down in this process and one has the following change of weights $[[w_1, 0, w_2]] \rightarrow [[w_1 + w_2, 0, 0]]$.*

(2) *Let $C_1 + C_2 + \dots + C_n$ be a chain (resp. a circular graph) with weights $0, 0, w_3, \dots, w_n$. Then there exists elementary transformations such that C_i, \dots, C_n are not blown down (where $3 \leq i \leq n$) and one has the following change of weights*

$$[[0, 0, w_3, \dots, w_n]] \rightarrow [[w_3, \dots, w_{i-2}, 0, 0, w_i, \dots, w_n]]$$

$$(resp. ((0, 0, w_3, \dots, w_n)) \rightarrow ((w_3, \dots, w_{i-2}, 0, 0, w_i, \dots, w_n))).$$

We shall use also the notion of a standard graph. In the case of chains example of standard graphs are given by $[[0]]$, $[[w_1, \dots, w_n]]$, and $[[0, 0, w_1, \dots, w_{n-1}]]$, where $n \geq 1$ and every $w_i \leq -2$.

We shall need later the following consequence of [8, Theorems 2.15 and Theorem 3.1].

Proposition 2.4. *Let Γ_i , $i = 1, 2$ be minimal graphs, $Br(\Gamma_i)$ be the set of branch points of Γ_i , and Γ_2 admit a reconstruction from Γ_1 . Then*

- (i) *$Br(\Gamma_i)$ is an invariant of the reconstruction, i.e., none of the vertices of $Br(\Gamma_1)$ is blown down in this procedure and they are transformed bijectively onto $Br(\Gamma_2)$ with preservation of valency;*
- (ii) *there is a bijection between connected components of $\Gamma_1 \ominus Br(\Gamma_1)$ and of $\Gamma_2 \ominus Br(\Gamma_2)$ such that every connected component Γ_2^0 of $\Gamma_2 \ominus Br(\Gamma_2)$ is obtained from the corresponding connected component Γ_1^0 of $\Gamma_1 \ominus Br(\Gamma_1)$ by a sequence of blowing up and blowing down;*
- (iii) *every minimal weighted graph Γ_1 can be reconstructed into some minimal weighted graph Γ_2 such that each connected component of $\Gamma_2 \ominus Br(\Gamma_2)$ is a standard graph;*
- (iv) *if Γ_1^0 and Γ_2^0 are standard graphs then the reconstruction of Γ_1^0 into Γ_2^0 can be achieved by elementary transformations.*

Since chains $[[w_1, \dots, w_n]]$ with every $w_i \leq -2$ do not admit nontrivial elementary transformations we have the following.

Corollary 2.5. *Let $\Gamma_i, Br(\Gamma_i)$, and Γ_i^0 be as in Proposition 2.4. Suppose that every weight in Γ_1^0 is at most -2. Then any relatively minimal reconstruction⁴ of Γ_1 into Γ_2 induces an identical transformation of Γ_1^0 into Γ_2^0 . In particular when every weight in the graph $\Gamma_1 \ominus Br(\Gamma_1)$ is at most -2 any relatively minimal reconstruction is the identical transformation of Γ_1 into Γ_2 .*

3. PRELIMINARIES ABOUT FOLIATIONS

A more detailed exposition of the results from this subsection can be found in [3], [4], or [5].

Definition 3.1. (1) A foliation \mathcal{F} on a smooth complex surface \bar{X} is given by an open covering $\{U_j\}$ of \bar{X} and holomorphic vector fields $\nu_j \in H^0(U_j, T\bar{X})$ with isolated zeros such that

$$\nu_i = g_{ij}\nu_j \text{ on } U_i \cap U_j$$

for invertible holomorphic functions $g_{ij} \in H^0(U_i \cap U_j, \mathcal{O}_X^*)$ where \mathcal{O}_X^* is the sheaf of invertible functions. Gluing orbits of $\{\nu_j\}$ one gets leaves of the foliation \mathcal{F} . The singular set $\text{Sing}(\mathcal{F})$ is the discrete subset of \bar{X} whose intersection with each U_j coincides with zeros of ν_j . The cocycle $\{g_{ij}\}$ define a holomorphic line bundle $K_{\mathcal{F}}$ which is called the canonical bundle of the foliation \mathcal{F} .

(2) This definition can be extended to the case of \bar{X} with cyclic quotient singularities only. That is, a singular point p of \bar{X} is locally of form $\mathbb{B}^2/\mathbb{Z}_k$ where \mathbb{B}^2 is a ball in \mathbb{C}^2 equipped with a linear \mathbb{Z}_k -action. In this case \mathcal{F} is defined as a foliation on $\bar{X} \setminus \text{Sing}(\bar{X})$ with the additional requirement the lifted foliation to $\mathbb{B}^2 \setminus \{(0,0)\}$ can be extended to a foliation on \mathbb{B}^2 with a non-vanishing associated vector field ν (and this must be true for any singular point of \bar{X}). We express this by saying

$$\text{Sing}(\bar{X}) \cap \text{Sing}(\mathcal{F}) = \emptyset.$$

⁴That is, a reconstruction without unnecessary blowing up.

Then $K_{\mathcal{F}}$ on \bar{X} is the direct image of the canonical bundle on $\bar{X} \setminus \text{Sing}(\bar{X})$ under the inclusion morphism $\bar{X} \setminus \text{Sing}(\bar{X}) \hookrightarrow \bar{X}$ (in this situation $K_{\mathcal{F}}$ is not a bundle but only a \mathbb{Q} -bundle).

(3) Foliation \mathcal{F} is called nef if $K_{\mathcal{F}}$ is nef.

(4) A singularity $p \in \text{Sing}(\mathcal{F})$ is reduced if the linear part of the corresponding vector field at p has eigenvalues λ_1, λ_2 such that either they are nonzero and $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$ or $\lambda_1 \neq 0 = \lambda_2$. In the former case p is called nondegenerate, in the latter case a saddle-node. The foliation \mathcal{F} is called reduced if all of its singularities are reduced. A theorem of Seidenberg [32] says that for any foliation there is a resolution $\pi : \hat{X} \rightarrow \bar{X}$ of singularities of \mathcal{F} such that the induced foliation $\hat{\mathcal{F}}$ on \hat{X} is reduced.

(5) The Kodaira dimension $\text{kod}(\mathcal{F})$ of a reduced foliation \mathcal{F} on a projective surface \bar{X} is the Kodaira-Iitaka dimension of its canonical bundle $K_{\mathcal{F}} \in \text{Pic}(\bar{X}) \otimes \mathbb{Q}$. That is,

$$\text{kod}(\mathcal{F}) = \limsup_{n \rightarrow +\infty} \frac{\log \dim H^0(\bar{X}, K_{\mathcal{F}}^{\otimes n})}{\log n}.$$

The next result describes McQuillan's contraction (e.g., see [3] or [4]).

Theorem 3.2. *Let $\hat{\mathcal{F}}$ be a reduced foliation on a projective surface \hat{X} with at most cyclic quotient singularities such that $\hat{\mathcal{F}}$ is not a fibration $\hat{X} \rightarrow B$ over a complete curve B with general fibers isomorphic to \mathbb{P}^1 . Then there exists a birational morphism $(\hat{X}, \hat{\mathcal{F}}) \rightarrow (\hat{X}', \hat{\mathcal{F}}')$ such that \hat{X}' is still projective with at most cyclic quotient singularities, $\hat{\mathcal{F}}'$ is still reduced, and $K_{\hat{\mathcal{F}}'}$ is nef.*

Remark 3.3. ([5, Section 3] or [4, page 10]) Contraction of \hat{X} to \hat{X}' is a sequence of blowing down of rational curves such that each of them is invariant with respect to the consequent induced foliation and the restriction of the canonical bundle of the foliation to the curve is negative. Every of these curves F contains exactly one singularity p of the foliation which is automatically a regular point of the surface. Furthermore, F is contracted to a point which is a regular point of the induced foliation on the resulting surface (but not in general a regular point of the surface).

Another result of MCQUILLAN crucial for us says the following (see [26, Sections IV and V] or [3, Chapter 9, Theorems 1 and 4, Corollary 1]).

Theorem 3.4. *Let $\hat{\mathcal{F}}$ be a reduced foliation on a smooth projective surface \hat{X} such that $\hat{\mathcal{F}}$ possesses a tangent entire curve that is Zariski dense in \hat{X} . Then the Kodaira dimension $\text{kod}(\hat{\mathcal{F}})$ is either 0 or 1. Furthermore,*

(1) *If $\text{kod}(\hat{\mathcal{F}}) = 1$ then $\hat{\mathcal{F}}$ is a Riccati foliation (or a Turbulent foliation), i.e. there exists a so called adapted (with respect to $\hat{\mathcal{F}}$) fibration $f : \hat{X} \rightarrow B$ whose general fiber is a rational curve (or an elliptic curve) transverse to \mathcal{F} .*

(2) *If $\text{kod}(\hat{\mathcal{F}}) = 0$ and $\theta : \hat{X} \rightarrow \hat{X}'$ is the MCQUILLAN's contraction to a nef reduced foliation $\hat{\mathcal{F}}'$ on \hat{X}' then there exists a finite covering $r : Y \rightarrow \hat{X}'$ such that*

(2a) *Y is smooth and r is ramified only over the quotient singularities of \hat{X}' .*

(2b) *The canonical bundle $K_{\mathcal{G}}$ of the lifted foliation $\mathcal{G} = r^*(\hat{\mathcal{F}}')$ is trivial, i.e. $K_{\mathcal{G}} = \mathcal{O}_Y$, and so \mathcal{G} is generated by a global holomorphic vector field with isolated zeros only.*

Notation 3.5. Let X be a smooth open algebraic surface equipped with a complete algebraic vector field ν and an SNC-completion \bar{X} (i.e. $\bar{D} = \bar{X} \setminus X$ is an SNC-divisor

in the smooth projective surface \bar{X}). The algebraic vector field ν generates a foliation \mathcal{F} on X which extends to \bar{X} . By $\pi : \hat{X} \rightarrow \bar{X}$ we denote the composition of a minimal sequence of blowing up such that \mathcal{F} induces a foliation $\hat{\mathcal{F}}$ on \hat{X} with all singularities reduced.

Of course, in this case $\hat{\mathcal{F}}$ admits a lot of entire tangent curves $\mathbb{C} \rightarrow \bar{X}$ but it may happen that none of them is Zariski dense in \hat{X} which is described by the following well-known fact.

Proposition 3.6. *Let Notation 3.5 hold and every general entire curve tangent to $\hat{\mathcal{F}}$ be algebraic. Then ν admits a rational first integral (i.e. a non-constant rational map $X \dashrightarrow B$ into a complete curve B with ν tangent to the fibers of this morphism).⁵*

Proof. By assumption general leaves of $\hat{\mathcal{F}}$ are complete algebraic curves in \hat{X} and they do not meet each other since every reduced singularity has locally at most two invariant curves through it (so-called strong and weak separatrices). Furthermore, in the reduced case the only singularities of leaves are normal crossings [4, page 7]. Therefore every general leaf C is isomorphic to \mathbb{P}^1 since it contains an integral curve of ν (isomorphic to \mathbb{C}^* or \mathbb{C}). Since general leaves belong to a smooth family of disjoint curves we have $C^2 = 0$. Hence by [1, Proposition 4.3] (which for convenience of readers is formulated below as Theorem 6.2) the complete linear system of C yields a morphism $\varphi : \hat{X} \rightarrow B$ which induces the desired rational first integral for ν on X . \square

We shall use also the following important result of Suzuki [35].

Theorem 3.7. *Let \mathcal{F} be a foliation on a normal Stein surface X .*

(1) *If all leaves of \mathcal{F} are properly embedded in $X \setminus \text{Sing}(\mathcal{F})$ then there is a non-constant meromorphic first integral of \mathcal{F} on X and a holomorphic map $\rho : X \setminus \text{Sing}(\mathcal{F}) \rightarrow R$ into a Riemann surface R such that*

(1a) *The irreducible components of the fibers of ρ are the leaves of \mathcal{F} .*

(1b) *The union $E \subset X$ of all reducible fibers of ρ has zero logarithmic capacity.*

(2) *Furthermore, if the general leaf of \mathcal{F} is isomorphic to \mathbb{C}^* (we shall call below such foliations of \mathbb{C}^* -type) then every leaf is closed in $X \setminus \text{Sing}(\mathcal{F})$ and therefore there is a meromorphic first integral as in (1).*

Definition 3.8. Recall that a semi-affine surface is an algebraic surface that admits a proper birational morphism onto an affine algebraic surface (which in complex analysis is nothing but the Remmert reduction). For instance, resolution of singularities of a normal affine surface S leads to a smooth semi-affine surface X . Algebraic vector fields on S lift to algebraic vector fields on X and complete ones to complete ones. Moreover vector fields on X are tangential to the preimage of the singularities of the surface thus they correspond to vector fields on S . Therefore to consider complete algebraic vector fields on normal affine surfaces is the same as to consider such fields on smooth semi-affine surfaces.

Remark 3.9. The Suzuki theorem remains valid with some adjustments in the case when \mathcal{F} is a foliation associated with a complete vector field ν on a normal semi-affine

⁵In the case of X isomorphic to \mathbb{C}^2 this follows from the classical Darboux theorem.

surface X . Indeed, let U be the union of complete curves contained in X and the zeros of ν . Since the set of such curves is discrete U must be invariant under the action of the flow of ν . Hence ν induces a complete vector field ν_0 in the Remmert reduction X_0 of X with the image U_0 of U playing the role of zeros of ν_0 . Hence the theorem holds with $X \setminus \text{Sing}(\mathcal{F})$ replaced by $X \setminus U \simeq X_0 \setminus U_0$.

Suppose that C is an \mathcal{F} -invariant curve. Then for every $p \in C$ one can define the Camacho-Sad index $\text{CS}(\mathcal{F}, C, p)$ (e.g., see [4]). This index vanishes when p is a regular point of the foliation. The following Camacho-Sad formula will be needed later.

Proposition 3.10. *In the notation as before one has*

$$C \cdot C = \text{CS}(\mathcal{F}, C) := \sum_{p \in \text{Sing}(\mathcal{F}) \cap C} \text{CS}(\mathcal{F}, C, p).$$

In particular the selfintersection of C is zero if there are no singularities of \mathcal{F} on C .

4. RICATTI FOLIATION IN THE CASE OF KODAIRA DIMENSION ONE

Proposition 4.1. *Let ν be a complete vector field on a smooth semi-affine surface X without rational first integral and let Notation 3.5 hold. Suppose that the induced foliation $\hat{\mathcal{F}}$ on \hat{X} is reduced and has Kodaira dimension one, i.e. by Theorem 3.4 (1) there exists an adapted fibration $f : \hat{X} \rightarrow B$ associated with $\hat{\mathcal{F}}$.*

Then B is \mathbb{P}^1 and the foliation $\hat{\mathcal{F}}$ is Riccati, i.e. the morphism $f : \hat{X} \rightarrow B$ is a \mathbb{P}^1 -fibration. Moreover if $\hat{D} = \pi^{-1}(\bar{X} \setminus D)$ then the restriction of f to $\hat{X} \setminus \hat{D}$ is a regular function that factors through a regular function $g : X \rightarrow \mathbb{C}$ with general fibers isomorphic to \mathbb{C} or \mathbb{C}^ such that the flow of ν sends fibers of g to fibers of g .*

Proof. This proposition was proven in [5] for the case $X = \mathbb{C}^2$ but actually the proof works in a more general setting. The first step is to exclude Turbulent foliations. Brunella proved in [5, Lemma 1] that the case of Turbulent foliations does not appear for \mathbb{C}^2 , but he only used the fact that the surface does not contain infinite numbers of elliptic curves which clearly holds for any semi-affine surface. Now suppose that B has genus ≥ 2 then there are only constant morphisms from \mathbb{C} to B . That is, every leaf of the foliation must be contained in a fiber of f contrary to the fact that such general leaves must be transversal to the foliation.

By the same reason in the case of a toric B there are no fibers of f tangent to $\hat{\mathcal{F}}$ (since otherwise one would have a non-constant holomorphic map from \mathbb{C} to a punctured torus). It is shown in [5, page 439] that the degree of the line bundle $f_*(K_{\hat{\mathcal{F}}})$ on the curve B is positive in case of Kodaira dimension 1. Furthermore such a foliation induces an orbifold structure on B and this degree coincides with the expression of form $-e_{\text{orb}}(B) + a$ where $e_{\text{orb}}(B)$ is the orbifold Euler characteristic and a is the sum of rational numbers assigned to fibers of f tangent to $\hat{\mathcal{F}}$ [5, page 439]. Since there are no fibers tangent to $\hat{\mathcal{F}}$ we have $a = 0$ and thus the inequality $e_{\text{orb}}(B) < 0$. Recall that

$$e_{\text{orb}}(B) = e_{\text{top}}(B) - \sum_{j=1}^k \left(1 - \frac{1}{m_j}\right)$$

where $e_{\text{top}}(B)$ is the topological Euler characteristic (i.e. it is zero since B is a torus), the sum runs over all singular points of B (with respect to the orbifold structure), and

m_j is their order. Since there is no nontrivial orbifold structure on the torus (e.g., [31]), $k = 0$ and we have $e_{\text{orb}}(B) = 0$ which is the desired contradiction.

Thus we have established the fact that the base B is \mathbb{P}^1 and the foliation is Riccati. Brunella proved in [5, pages 439-441] the existence of the function g in the last statement of the Proposition for the case \mathbb{C}^2 , but, again, the proof works for all rational semi-affine surfaces. \square

Corollary 4.2. *Under the assumption of Proposition 4.1 the surface X is rational.*

5. RICCATI FOLIATION IN THE CASE OF KODAIRA DIMENSION ZERO

Notation 5.1. In this section X is a smooth semi-affine surface and we keep Notation 3.5 for symbols $\nu, \bar{X}, \bar{D}, \mathcal{F}, \hat{X}, \hat{\mathcal{F}}$, and $\pi : \hat{X} \rightarrow \bar{X}$. Note that $\hat{X} \setminus \hat{D}$ is still semi-affine where $\hat{D} = \pi^{-1}(\bar{D})$ and ν induces a complete algebraic vector field $\hat{\nu}$ on $\hat{X} \setminus \hat{D}$ since each blowing-up used in the construction of \hat{X} occurs at a singularity of the corresponding foliation. In particular the flow of $\hat{\nu}$ is a holomorphic \mathbb{C}_+ -action on $\hat{X} \setminus \hat{D}$. By $\theta : \hat{X} \rightarrow \hat{X}'$ we denote the McQuillan contraction (i.e., \hat{X}' is normal with cyclic quotient singularities only). Since all curves contracted by θ are $\hat{\mathcal{F}}$ -invariant one has again the induced foliation $\hat{\mathcal{F}}'$ on \hat{X}' . Furthermore the Hartogs theorem implies that there is an induced \mathbb{C}_+ -action on $\hat{X}' \setminus \hat{D}'$ for $\hat{D}' = \theta(\hat{D})$ which is the flow of a complete field $\hat{\nu}'$. By $r : Y \rightarrow \hat{X}'$ we denote the finite morphism ramified only over the quotient singularities of \hat{X}' as in Theorem 3.4. That is, Y is a smooth projective surface for which the foliation \mathcal{G} induced by $\hat{\mathcal{F}}'$ is generated by a global holomorphic vector field μ on Y with at most isolated zeros. We also let $T = r^{-1}(\hat{D}')$ and $\tilde{\nu}$ be the pull back of $\hat{\nu}'$ to $Y \setminus T$ by r . That is, the vector field $\tilde{\nu}$ is still complete on $Y \setminus T$ and $\tilde{\nu} = f\mu$ for a rational function f on Y .

Remark 5.2. Since X is semi-affine, \bar{D} and \hat{D} are connected by the Lefschetz hyperplane section theorem and therefore, $\theta^{-1}(\theta(\hat{D})) = \hat{D} \cup A$ is connected by the Zariski connectedness theorem.

Lemma 5.3. *Let $\hat{D}' = \theta(\hat{D})$. Then*

- (i) \hat{D}' (and therefore T) is a divisor in \hat{X}' and $\hat{X}' \setminus \hat{D}'$ (resp. $Y \setminus T$) is semi-affine;
- (ii) general leaves of \mathcal{G} are biholomorphic to \mathbb{C} or \mathbb{C}^* ;
- (iii) μ has zeros.

Proof. If \hat{D}' is a singleton $p \in \hat{X}'$ then every holomorphic function on $\hat{X}' \setminus p$ (and therefore on $\hat{X} \setminus \hat{D}$) is constant which is not true. Thus \hat{D}' is a divisor. Furthermore, by the Nakai-Moishezon criterion (e.g., see [18]) the fact that $\hat{X} \setminus \hat{D}$ is semi-affine is equivalent to the fact that \hat{D} is a support of an effective divisor B such that for any irreducible curve C whose intersection with B is not empty one has $B \cdot C > 0$. Connectedness of $\hat{D} + A$ and the standard argument involving induction on the number of components of A implies that $\hat{D} + A$ has the same property, i.e. $\hat{X} \setminus \theta^{-1}(\hat{D}')$ is semi-affine. Hence $\hat{X}' \setminus \hat{D}'$ is semi-affine which is (i).

The general leaves of $\hat{\mathcal{F}}'$ are biholomorphic to a rational curve since the foliation is induced by a complete vector field $\hat{\nu}'$ on the semi-affine variety $\hat{X}' \setminus \hat{D}'$. Since r is ramified over a finite number of points the same is true for general leaves of \mathcal{G} . Therefore, they are not elliptic curves and the possibilities left are \mathbb{C} and \mathbb{C}^* .

Assume that μ has no zeros and therefore \mathcal{G} has no singularities. Then in the case of \mathcal{G} -invariant T the Camacho-Sad formula in Proposition 3.10 implies that $T \cdot T = 0$ contrary to the fact that $Y \setminus T$ is semi-affine.

Thus the union T_0 of irreducible components of T that are not \mathcal{G} -invariant is not empty and therefore T meets a general leaf L of \mathcal{G} . Note that they meet precisely at one point and $L \simeq \mathbb{C}$ since otherwise $L \setminus T$ cannot be an integral curve of the complete vector field $\tilde{\nu}$ on $Y \setminus T$ from Notation 5.1. In particular, the general leaf L' of the foliation associated with $\tilde{\nu}$ in $Y \setminus T$ is \mathbb{C}^* and by the Suzuki theorem every leaf $L' \subset Y \setminus T$ of this foliation is properly embedded in $Y \setminus (T \cup U)$ where U is from Remark 3.9.

Let us show that U is empty. Assume that there is a complete curve C in $Y \setminus T$. Then C is fixed by the flow of μ since the image of this curve need to be a complete curve nearby. This means that C is invariant and thus $C \cdot C = 0$ by the absence of zeroes and the Camacho-Sad formula in Proposition 3.10. This is a contradiction since a curve of self-intersection zero cannot belong to a contractible set by the Grauert criterion (e.g. see [1]) for contractible curves. Thus U is empty (and in particular $Y \setminus T$ is affine).

Hence $L \setminus T$ is properly embedded in $Y \setminus T$ which implies that L is properly embedded in $Y \setminus T_1$ where T_1 is the union of \mathcal{G} -invariant components of T . Therefore the topological closure \bar{L} of L contains a point of a connected component T' of T_1 . Since μ has no zeros and \bar{L} is \mathcal{G} -invariant we have $T' \subset \bar{L}$. Note also that T' must meet T_0 since otherwise by the Camacho-Sad formula $T \cdot T' = T' \cdot T' = 0$ contrary to affineness. Let $x_0 \in T' \cap T_0$. Choose a neighborhood V of x_0 in T_0 such that \bar{V} is disjoint from $L \cap T_0$. Choose $\varepsilon > 0$ such that the flow map of μ with starting points in V gives a biholomorphism of a neighborhood Ω of x_0 in Y with $V \times \{z \in \mathbb{C} \mid |z| < \varepsilon\}$. If now y is a point in $L \cap \Omega$ by construction of Ω the leaf L meets V . Thus L meets T_0 in two different points, a contradiction. □

Notation 5.4. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric with coordinates (x, y) , C_0, C_1, C_2 and C_3 be the lines $\{x = \infty\}, \{y = \infty\}, \{x = 0\}$ and $\{y = 0\}$, and $\mu_0 = \mu_1 + \mu_2$ be a vector field on Q such that μ_i is a nonzero holomorphic on a factor \mathbb{P}^1 . In particular without loss of generality one can assume one of the following three choices for the set of zeros of μ_0 :

- (a) $C_0 \cap C_1 = (\infty, \infty)$;
- (b) $C_0 \cap C_1 = (\infty, \infty), C_1 \cap C_2 = (0, \infty)$; or
- (c) $C_0 \cap C_1 = (\infty, \infty), C_0 \cap C_3 = (\infty, 0), C_1 \cap C_2 = (0, \infty)$, and $C_2 \cap C_3 = (0, 0)$.

Up to a nonzero factor in (a) one can suppose that the vector field μ_0 is of form $\mu_0 = \partial/\partial x + \alpha \partial/\partial y$ where $\alpha \in \mathbb{C}^*$, in (b) $\mu_0 = x \partial/\partial x + \beta \partial/\partial y$, and in (c) $\mu_0 = x \partial/\partial x + \beta y \partial/\partial y$ with $\beta \in \mathbb{C}^*$.

We let also $Q^0 = Q \setminus (C_0 \cup C_1 \cup C_2) \simeq \mathbb{C}^* \times \mathbb{C}$ (resp. $Q^0 = Q \setminus (C_0 \cup C_1 \cup C_2 \cup C_3) \simeq \mathbb{C}^* \times \mathbb{C}^*$) in case (b) (resp. (c)).

Remark 5.5. Note that $\alpha x - y$ is a rational first integral in case (a). Similarly one has a rational first integral in case (c) when $\beta \in \mathbb{Q}$. If $\beta \notin \mathbb{Q}$ then in case (c) Q^0 does not contain algebraic curves tangent to μ_0 . More precisely the integral curves are of

form $(x_0 e^t, y_0 e^{\beta t})$ with nonzero x_0, y_0 , and t running over \mathbb{C} . The same is true in case (b) where the integral curves are of form $(x_1 e^t, y_1 + \alpha t)$ with $x_1 \in \mathbb{C}^*, y_1 \in \mathbb{C}$.

Lemma 5.6. *Suppose that a foliation \mathcal{F} on X has no rational first integral and $\text{kod}(\hat{\mathcal{F}}) = 0$ for the induced reduced foliation on \hat{X} . Then there is a vector field μ_0 as in (b) or (c) from Notation 5.4 and birational map $\chi : Y \dashrightarrow Q$ such that it transforms μ to μ_0 and such that it is an isomorphism over $Q \setminus \bigcup_{i \neq j} C_i \cap C_j$ (and in particular over Q^0).*

Proof. We need to go through the list of four possible projective smooth surfaces that admit holomorphic vector fields with isolated zeros [3, Section 6][4, page 13]. In the first case Y is an isotrivial fibration with all fibers being elliptic curves which serve also as leaves of the foliation \mathcal{G} . Hence we disregard this possibility by Lemma 5.3 (ii).

In the second case Y is a torus \mathbb{C}^2/G with μ induced by a constant vector field on \mathbb{C}^2 and in the third case Y is a \mathbb{P}^1 -fibration over an elliptic curve B and μ projects to a constant vector field on B . In both cases μ is nowhere vanishing and thus these cases do not occur by Lemma 5.3.

This leaves us with the fourth possibility which is exactly a birational map $\chi : Y \dashrightarrow Q$ such that it transforms μ to μ_0 from Notation 5.4. In combination with the fact that μ must be regular on Y this implies that χ is isomorphism over $Q \setminus \bigcup_{i \neq j} C_i \cap C_j$. \square

Lemma 5.7. *Let notation of Lemma 5.6 hold and let $\tilde{\nu}$ be the pull back of $\hat{\nu}'$ to $Y \setminus T$. Then in case (c) the image of $T \subset Y$ does not meet Q^0 and we have $\tilde{\nu} = \alpha \mu$ on $Y \setminus T$ for some $\alpha \in \mathbb{C}$. In case (b) either*

(b1) *the image of $T \subset Y$ does not meet Q^0 and again $\tilde{\nu} = \alpha \mu$, or*

(b2) *it meets Q^0 along a line $y = a$ and $\tilde{\nu} = \alpha(y - a)\mu$,*

for some $\alpha, a \in \mathbb{C}$ (where by abuse of notation we denote the lift-up of coordinate y to Y by the same symbol).

Proof. Remark 5.5 describes the general form of a leaf of \mathcal{G} which implies in particular that $T \cap Q^*$ is not \mathcal{G} -invariant where $Q^* = \chi^{-1}(Q^0)$. Recall that ν induces a complete vector field $\tilde{\nu}$ on $Y \setminus T$ whose integral curves are contained in the leaves of \mathcal{G} , i.e. $\tilde{\nu} = f\mu$ where f is a rational function on Q lifted to Y . By [5, Remark 1] $\tilde{\nu}$ must vanish on $T \cap Q^*$, i.e. $f = p(x, y)/(x^i y^j)$ with $i, j \geq 0$ in case (c) and $f = p(x, y)/x^i$ in case (b) where $p(x, y)$ is a polynomial.

Present a general leaf F of \mathcal{G} as a curve parametrized by $t \in \mathbb{C}$ according to Remark 5.5 and note that the restriction of $\tilde{\nu}$ to F is complete if and only if $\tilde{\nu}|_F$ is of form $g(t)\partial/\partial t$ where $g(t) = at + b$ is a linear polynomial. That is, in case (c) we must have an equality

$$p(x_0 e^t, y_0 e^{\beta t}) e^{-(i+\beta j)t} = x_0^i y_0^j (at + b).$$

Recall that β is irrational. Hence if p is not a monomial the left-hand side of the last equality is a sum of more than one exponents with different powers, i.e. it cannot be equal to a linear polynomial. Thus p is monomial and furthermore up to a constant factor $p(x, y) = x^i y^j$ and $a = 0$. This yields case (c).

In case (b) the similar argument implies that $f = p(x, y)/x^i$ is a polynomial in y only and furthermore this polynomial is linear which concludes the proof.

□

Remark 5.8. By a change of coordinate, in case (b2), we can also suppose that $a = 0$. Then the image of T is still contained in $\bigcup_{i=0}^3 C_i$ even if C_3 is not \mathcal{G} -invariant in this case.

Proposition 5.9. *Let ν be a complete algebraic vector field on a smooth semi-affine surface X which has no rational first integral and the Kodaira dimension of its induced foliation $\hat{\mathcal{F}}$ is zero. Then the variety \hat{X}' is smooth, i.e. $r : Y \rightarrow \hat{X}'$ is an isomorphism.*

Proof. We have to consider the cases (b1), (b2) and (c) from Lemma 5.7.

In the cases (b1) and (c) $\tilde{\nu}$ is a constant multiple of μ on $r^{-1}(\hat{X}' \setminus \hat{D}')$. Thus $\tilde{\nu} = r^*(\hat{\nu}')$ extends regularly to Y and $\hat{\nu}'$ extends regularly to \hat{X}' . Assume that $x_0 \in \hat{X}'$ is a cyclic quotient singularity as in Definition 3.1 (2) and $y_0 \in r^{-1}(x_0)$. That is, in a coordinate neighborhood of y_0 the map r can be viewed as a quotient morphism with respect to an action of a finite cyclic group G for which y_0 is fixed and the origin is the only fixed point for the induced G -action on the tangent space $T_{y_0}Y$. By construction $\tilde{\nu}$ is invariant with respect to the G -action. Hence $\tilde{\nu}(y_0) = 0$. On the other hand μ cannot vanish at y_0 since any quotient singularity of the resulting variety \hat{X}' in the McQuillan contraction is not a singularity of the foliation $\hat{\mathcal{F}}'$ [4]. Hence we get a desired contradiction.

In case (b2) consider $A = Y \setminus Q^*$ where as before $Q^* = \chi^{-1}(Q^0)$. By abuse of notation we still use the same symbols x and y to denote a coordinate system on $Q^* \simeq Q^0$ and we let C'_i be the proper transform of C_i in Y under χ . Note that A is r -saturated (i.e. $r^{-1}(r(A)) = A$) since A is the union of all \mathcal{G} -invariant algebraic curves in Y . The same holds for $C'_3 = \{y = 0\}$ since it is the zero divisor of $\tilde{\nu} = y\mu$ and for the set $\text{Sing}(\mathcal{G})$.

We claim that the set $C'_0 \cup C'_2$ is also r -saturated. Indeed let $C \subset r^{-1}(r(C'_0 \cup C'_2))$ be an irreducible curve then $r(C) \cap r(C'_3) \neq \emptyset$ and thus, since C'_3 is r -saturated, we have $C \cap C'_3 \neq \emptyset$ which only leaves the possibility $C = C'_0$ or $C = C'_2$. Therefore the intersection $\text{Sing}(\mathcal{G}) \cap (C'_0 \cup C'_2)$ is r -saturated, moreover this set consists of the two points p_0 and p_2 which are the intersection points of C'_i with $\chi^{-1}(\{y = \infty\})$. Since $r(p_0)$ and $r(p_2)$ are singularities of the foliation they are smooth points of the surface \hat{X}' (see Remark 3.3) and thus no ramification points.

Therefore the intersection $\text{Sing}(\mathcal{G}) \cap (C'_0 \cup C'_2)$ is r -saturated. Moreover this set consists of the two points p_0 and p_2 which are the intersection points of C'_0 and C'_2 respectively with $\chi^{-1}(\{y = \infty\})$. Since $r(p_0)$ and $r(p_2)$ are singularities of the foliation they are smooth points of the surface \hat{X}' (see Remark 3.3) and r is not ramified at them.

Therefore if r is not bijective it is a two-sheeted covering and hence r can be viewed as the quotient morphism of a \mathbb{Z}_2 -action on Y (switching points p_0 and p_2) such that $Y/\mathbb{Z}_2 = \hat{X}'$.

Since A is preserved by the action its restriction acts on $Q^* = \mathbb{C}_x^* \times \mathbb{C}_y$. By the fundamental theorem of algebra it yields a map of form $\varphi : (x, y) \rightarrow (\lambda x, f(x, y))$ where $\lambda \in \mathbb{C}^*$. Since the square of the map is identity, C_3 is preserved and the map has isolated fixed points it is given by $\varphi : (x, y) \rightarrow (-x, -y)$. However φ must preserve the foliation generated by the field $x\partial/\partial x + \alpha\partial/\partial y$ which it does not. A contradiction. □

Theorem 5.10. *Let ν be a complete algebraic vector field on a smooth semi-affine surface X which has no rational first integral and the Kodaira dimension of its induced foliation $\hat{\mathcal{F}}$ is zero, then*

- (i) *the surface X is rational;*
- (ii) *there is a regular function $f : X \rightarrow \mathbb{C}$ with general fibers isomorphic to \mathbb{C} or \mathbb{C}^* and such that the flow of ν sends every fiber of f to a fiber of f .*

Proof. By Remark 5.5 we have to deal with cases (b) and (c). Remark 5.8 and Propositions 5.9 show that there are birational morphisms $\theta : \hat{X} \rightarrow Y$ and $\chi : Y \rightarrow Q = \mathbb{P}_x^1 \times \mathbb{P}_y^1$ such that the pull back of ν to Q (via $\pi : \hat{X} \rightarrow \bar{X}$, θ , and χ) is given by μ_0 in cases (b1) and (c), and by $y\mu_0$ in case (b2). In particular we get a birational map $X \dashrightarrow Q$, i.e. X is rational which is (i).

For (ii) we first deal with the case (b2). Note that C_3 is the zero divisor of $y\mu_0$ on Q while C_1 is its polar divisor. Furthermore the flow of $y\mu_0$ transforms each fiber of y to a similar one. Because ν has no poles on X the preimage of C_1 in \hat{X} is contained in \hat{D} . Therefore the function $y \circ \chi \circ \theta$ is regular on $\hat{X} \setminus \hat{D}$. That is, its push-down f to X is the desired regular function.

From now on we use the notion of dual graphs Section 2. Let us consider the case (c). First we show that the dual graph of $\hat{D} \subset \hat{X}$ is circular. Indeed the field μ_0 is regular and it is non-zero at every point of Q different from any point of form $C_i \cap C_j$ for some $i \neq j$. Consider $\chi : Y \rightarrow Q$ that is a composition of blowing up. If the first of these blowing up occurs at a point where μ_0 is non-zero then the resulting field has a pole, i.e. the resulting field on Y has a pole contrary to the fact that μ is regular. Hence the first blow-up occurs at some $C_i \cap C_j$ and the preimage of the curve $Q \setminus Q^0$ has a circular dual graph. Using the form of μ_0 of Notation 5.4 one can check that μ_0 induces a reduced foliation on Q and such foliations stay reduced after blowing-ups of singular points (e.g. see [3] or [4]). Hence the field induced by μ_0 is nonzero at every point of the exceptional divisor E different from the two points where E meets the proper transforms of C_i and C_j (at these two points there are reduced singularities of the induced foliation). The argument as before shows that the next blowing up may occur only at a singularity of the foliation. Hence induction by the number of blowing up implies that the dual graph of $Y \setminus Q^*$ (where $Q^* = \chi^{-1}(Q^0)$) is circular. Presenting now $\chi : \hat{X} \rightarrow Y$ as a composition of blowing up, using again induction, and the argument as before we see that

- (i) the dual graph Γ of $\theta^{-1}(Y \setminus Q^*)$ contains a circular subgraph Γ_0 such that $\hat{\nu}$ is regular on any irreducible curve serving as a vertex in Γ_0 and
- (ii) any component R of $\Gamma \ominus \Gamma_0$ is contractible and $\hat{\nu}$ has poles on any irreducible curve C corresponding to a vertex of R .

Since $\hat{\nu}$ is regular on $\hat{X} \setminus \hat{D}$ we have $C \subset \hat{D}$. Thus contracting such curves we can suppose that $\Gamma = \Gamma_0$ and $\hat{\nu}$ is regular on \hat{X} . Furthermore the classification of circular graphs in [8, Section 2.4] and the fact that $\hat{X} \setminus \hat{D}$ is semi-affine implies that after reconstruction one can suppose that one of irreducible components C of \hat{D} has $C^2 = 0$. That is, C is a fiber of a \mathbb{P}^1 -fibration by Theorem 6.2 sd [1, page 142] whose restriction to $\hat{X} \setminus \hat{D}$ yields a regular function f with general \mathbb{C}^* -fibers. Since the flow

of $\hat{\nu}$ preserves C it preserves the \mathbb{P}^1 -fibration, i.e. it transforms each fiber of f into a similar fiber and we are done.

In case (b1) the dual graph is linear and the construction of the regular function is similar. □

Remark 5.11. Proposition 4.1 and Theorem 5.10 show that for every nontrivial complete algebraic vector field ν on X without rational first integral one has a \mathbb{C} or \mathbb{C}^* -fibration $f : X \rightarrow \mathbb{C}$ such that for a complete nontrivial vector field ν_0 on \mathbb{C} this map f is \mathbb{C}_+ -equivariant with respect to holomorphic \mathbb{C}_+ -action on X (resp. on \mathbb{C}) generated by ν (resp. ν_0). In particular the phase flow of ν transforms every general fiber of f into a different general fiber of f . However if a fiber $f^{-1}(b)$ is singular (i.e. f is not locally trivial in any neighborhood U of $b \in \mathbb{C}$)⁶ then this phase flow preserves $f^{-1}(b)$ since it preserves local triviality.

6. \mathbb{P}^1 -FIBRATIONS

The results of Section 4 and 5 suggest that in order to classify complete algebraic vector fields on a semi-affine smooth surface X we need to understand fibrations of X with general fibers \mathbb{C} or \mathbb{C}^* , for short \mathbb{C} - or \mathbb{C}^* -fibrations. They can be extended to \mathbb{P}^1 -fibrations (i.e. fibrations with general fiber isomorphic to \mathbb{P}^1) of a smooth completion \bar{X} of X . Hence in this section we present some general results on \mathbb{P}^1 -fibrations.

Notation 6.1. For the rest of the section we suppose that $\bar{f} : \bar{X} \rightarrow B$ is a \mathbb{P}^1 -fibration of a smooth projective surface over a smooth complete curve. Let \bar{D} be a connected curve in \bar{X} of simple normal crossing (SNC) type and $X = \bar{X} \setminus \bar{D}$. We suppose that $f = \bar{f}|_X$ is a \mathbb{C} or \mathbb{C}^* -fibration on X .

A classical result about \mathbb{P}^1 -fibrations is the following, see Proposition 4.3 in [1]:

Theorem 6.2. *Let \bar{X} be a smooth compact surface and C be smooth rational curve on \bar{X} . If $C^2 = 0$, then there exists a sequence of blowing up $\varphi : X \rightarrow Y$, where Y is ruled (a \mathbb{P}^1 -bundle over a curve), such that C meets no exceptional curve of φ , and $\varphi(C)$ is a general fiber of Y .*

In particular this theorem states that singular fibers of a \mathbb{P}^1 -fibration are contractible to a rational curve and thus their dual graphs do not contain cycles. The following lemma gives a slightly more precise statement about the singular fibers of a \mathbb{P}^1 -fibration.

Lemma 6.3. *Let Notation 6.1 hold and Γ be the dual graph of a fiber $F = \bar{f}^{-1}(b)$ for some $b \in B$.*

(1) *Suppose that E is a component of F that is reduced in $\bar{f}^*(b)$. Then $\Gamma \ominus E$ is contractible and furthermore after this contraction the weight of E becomes 0.*

(2) *Let E_1 and E_2 be vertices of Γ that are reduced in $\bar{f}^*(b)$ and let Γ^0 be the smallest linear subgraph of Γ containing E_1 and E_2 . Then $\Gamma \ominus \Gamma^0$ is contractible.*

⁶It is known that for \mathbb{C}^* -fibrations (resp. \mathbb{C} -fibrations) a fiber is non-singular iff it is reduced and isomorphic to \mathbb{C}^* (resp. \mathbb{C}).

Proof. The first statement can be found, say, in [27, Lemma 2.11.2]. For the second statement we contract all components of $\Gamma \ominus E_1$ that do not contain E_2 and vice versa. Assume that $\Gamma \ominus \Gamma_0$ is still not empty and it does not contain linear (-1) -vertices. It is enough to show that this is impossible.

Since any connected components of $\Gamma \ominus E_1$ must be contractible by (1) this assumption implies that E_1 and similarly E_2 are end-vertices of Γ . Thus if $\Gamma \neq \Gamma^0$ there exists a vertex $E \in \Gamma^0 \ominus (E_1 \cup E_2)$ such that it is a branch point of Γ . Since the graph is contractible to E_1 the component of $\Gamma \ominus E$ containing E_2 must be contractible to a point in the curve E . This implies that E is reduced in $\bar{f}^*(b)$ (indeed, if E has multiplicity at least 2 so does E_2 because E_2 is obtained from a point in E by a sequence of blow-ups). By (1) all components of $\Gamma \ominus E$ are contractible contradicting the assumption that $\Gamma \ominus \Gamma^0$ does not contain (-1) vertices. \square

Definition 6.4. If $f : X \rightarrow B$ is a \mathbb{C}^* -fibration then \bar{D} either contains two sections B_1 and B_2 of \bar{f} (we call this case untwisted) or it contains a curve B_0 such that $\bar{f}|_{B_0} : B_0 \rightarrow B$ is a ramified double cover of rational curves (so-called twisted case).

Remark 6.5. It is worth mentioning that in the twisted case a fiber $\bar{f}^{-1}(b), b \in B$ meets B_0 at one point if and only if b is a ramification point of morphism $\bar{f}|_{B_0} : B_0 \rightarrow B$. The number of ramifications points is determined by the Riemann-Hurwitz formula and when $B_0 \simeq \mathbb{P}^1$ there are exactly two of them.

Proposition 6.6. *Let Notation 6.1 hold and $F = \bar{f}^{-1}(b)$ be a fiber contained in \bar{D} . Suppose that the dual graph Γ of \bar{D} does not contain linear (-1) -vertices different from irreducible curves on which the restriction of \bar{f} is non-constant (we call such \bar{D} pseudo-minimal). Let Γ_0 be the smallest subgraph of Γ that contains all components of F and their neighbors.*

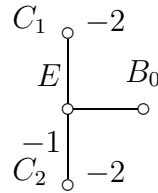
(1) *If f is a \mathbb{C} -fibration then F is a 0-vertex and Γ_0 is a linear chain $F + B'$ where B' is a section of \bar{f} .*

(2) *If f is an untwisted \mathbb{C}^* -fibration then F is a 0-vertex and Γ_0 is a linear chain $B_1 + F + B_2$.*

(3) *Let f be a twisted \mathbb{C}^* -fibration.*

(3a) *Suppose that F meets B_0 at two points. Then F is a 0-vertex and Γ_0 is a cycle consisting of F , and B_0 joined by two edges.*

(3b) *Suppose that F meets B_0 at one point. Then F is a linear chain $C_1 + E + C_2$ where C_1 and C_2 are (-2) -vertices, E is a (-1) -vertex, and $\Gamma_0 \ominus E$ contains three components C_1, C_2 , and B_0 . That is, the following*



is the form of Γ_0 (in the rest of the paper subgraphs of Γ satisfying the assumptions of (3b) will be called subgraphs of type Γ_*).

Proof. Note that in (1) B' is the only neighbor of F in Γ since otherwise f is not a \mathbb{C} -fibration. The component E of F that meets B' is reduced in $\bar{f}^*(b)$ because $f^*(b) \cdot B' = 1$. By Lemma 6.3 one can contract F to E but since F does not have linear (-1) -vertices we have $F = E$.

Since f is a \mathbb{C}^* fibration in (2), by the similar reason B_1 and B_2 are the only neighbors of F in Γ and the component E_i of F that meets B_i is reduced in $\bar{f}^*(b)$. By Lemma 6.3 and pseudo-minimality F must be a linear chain with E_1 and E_2 being end vertices. In particular E_1 and E_2 are linear vertices of Γ . Note that if $E_1 \neq E_2$ then both of them are (-1) -curves since the curve $\overline{F \setminus E_i}$ is contractible by Lemma 6.3. This contradicts pseudo-minimality. Thus $E_1 = E_2 = F$ which yields (2).

In (3a) as before we have B_0 as the only neighbor of F in Γ and the neighbors of B_0 in F are E_1 and E_2 which are reduced in $\bar{f}^*(b)$ (indeed we have $f^*(b) \cdot B_0 = 2$). Again, by Lemma 6.3 and pseudo-minimality we see that $F = E_1 = E_2$, i.e. being an irreducible fiber of \mathbb{P}^1 -fibration F is the 0-vertex which is (3a).

In (3b) B_0 is again the only neighbor of F in Γ . More precisely B_0 is a neighbor of vertex E in the graph Γ_1 of F and E has multiplicity 2 in $\bar{f}^*(b)$ since for the double section B_0 one has $\bar{f}^*(b) \cdot F = 2$. Since Γ_1 is contractible to a 0-vertex it contains a linear (-1) -vertex. The assumption on (-1) -vertices implies that this vertex is not in $\Gamma_1 \ominus E$. Hence E is a linear (-1) -vertex in Γ_1 . Let C_1 and C_2 be its neighbors in Γ_1 . Note that they are reduced in $\bar{f}^*(b)$. Indeed, since E is contractible to the points of intersection of the images of C_1 and C_2 the multiplicity of C is the sum of multiplicities of C_1 and C_2 , i.e. $2=1+1$. By Lemma 6.3 one can contract all components of $\Gamma_1 \ominus (C_1 \cup E \cup C_2)$ and we are done. \square

Remark 6.7. Let C be a smooth rational curve in \bar{X} with $C^2 = 0$ (i.e. it is a fiber of a \mathbb{P}^1 -fibration by Theorem 6.2). The following converse of Proposition 6.6 is true.

(1) Let C be an end vertex of Γ , then X admits a \mathbb{C} -fibration f such that C is a fiber of \bar{f} .

(2) Let C be a linear vertex of Γ with two distinct neighbors B_1 and B_2 . Then X admits an untwisted \mathbb{C}^* -fibration f such that C is a fiber of \bar{f} and B_1 and B_2 are sections of \bar{f} .

(3a) Let C be a linear vertex of Γ with one neighbor B_0 only (i.e. C is joined with B_0 by two edges). Then X admits a twisted \mathbb{C}^* -fibration f such that C is a fiber of \bar{f} and B_0 is the double section \bar{f} which intersects C transversally in two points.

(3b) Let $C_1 + C + C_2$ be a linear subgraph of Γ as in (3b) with B_0 being the only neighbor of C different from C_1 and C_2 . Then X admits a twisted \mathbb{C}^* -fibration f such that $C_1 \cup C \cup C_2$ is a singular fiber of \bar{f} and $B_0 \subset \bar{D}$ is the double section of \bar{f} .

We need one more technical fact for Section 9.

Lemma 6.8. *Let Notation 6.1 hold, $\bar{f} : \bar{X} \rightarrow B$ be a pseudo-minimal extension of f , and U be the union of components of \bar{D} on which the restriction of \bar{f} is not constant. Suppose that E_1 and E_2 are the only irreducible components of $\bar{f}^{-1}(b)$, $b \in B$ that meet the divisor U (where we allow equality $E_1 = E_2$) and that Γ^0 is as in Lemma 6.3.*

- (i) Let F be an irreducible affine⁷ component of $f^{-1}(b)$ whose closure is not a vertex in Γ^0 . Then $F \simeq \mathbb{C}$.
- (ii) Let Γ' be the dual graph of $\bar{f}^{-1}(b) \cup U$. Suppose that an irreducible component $V \subset \bar{f}^{-1}(b) \cap \bar{D}$ is a branch point of Γ' of valency m . Then (a) in the case when f is a \mathbb{C} -fibration or $V \notin \Gamma^0$ there are at least $m - 1$ components of $f^{-1}(b)$ isomorphic to \mathbb{C} (b) while the number of such components is at least $m - 2$ when f is a \mathbb{C}^* -fibration and $V \in \Gamma^0$.

Proof. Statement (i) is true since otherwise either the dual graph of $\bar{f}^{-1}(b)$ contains a cycle (contrary to the fact that the graph of $\bar{f}^{-1}(b)$ can be contracted to a 0-vertex) or the intersection of $\bar{f}^{-1}(b)$ with U is at least 2 (resp. 3) when f is a \mathbb{C} -fibration (resp. \mathbb{C}^* -fibration) which is absurd.

Exactly by the same reason the number of connected components of $\overline{(\bar{f}^{-1}(b) \cup U)} \setminus V$ that do not contain irreducible components of U coincides with $m - 1$ in case (a) and with $m - 2$ in case (b).

Let \mathcal{C} be such a component. By Lemma 6.3 \mathcal{C} is contractible. By pseudo-minimality assumption one cannot contract first an irreducible component of $\mathcal{C} \cap \bar{D}$. Hence $\mathcal{C} \neq \mathcal{C} \cap \bar{D}$. Note that any connected component of $\overline{\mathcal{C} \setminus \bar{D}}$ meets \bar{D} at one point because of (i). Thus we have the desired conclusion. \square

7. RATIONAL FIRST INTEGRAL, I

Proposition 7.1. *Let B be a germ of a smooth curve at point o and $\mu = x\partial/\partial x$ be the vector field on $B \times \mathbb{P}_x^1$, i.e. the set of its zeros is the union of $B \times \{\infty\}$ and $B \times \{0\}$. Suppose that $\text{pr} : \tilde{X} \rightarrow B$ is a smooth \mathbb{P}^1 -fibration and $\psi : \tilde{X} \rightarrow B \times \mathbb{P}_x^1$ is a birational morphism over B whose restriction over $B \setminus o$ is an isomorphism. Let the dual graph of $\text{pr}^{-1}(o)$ be linear with endvertices meeting the proper transforms \tilde{B}_1 and \tilde{B}_2 of $B \times \{\infty\}$ and $B \times \{0\}$ respectively. Then μ induces a regular complete vector field $\tilde{\mu}$ on \tilde{X} tangent to the fibers of pr and such that its restriction to $\text{pr}^{-1}(o)$ vanishes only at double points of the curve $\tilde{B}_1 \cup \text{pr}^{-1}(o) \cup \tilde{B}_2$.*

Proof. We use induction on the number k of irreducible components of $\text{pr}^{-1}(o)$. If $k = 1$ then ψ is an isomorphism and there is nothing to prove. Note that contracting a (-1) -curve in $\text{pr}^{-1}(o)$ we obtain a birational morphism $\sigma : \tilde{X} \rightarrow \check{X}$ over B such that ψ factors through it and the dual graph of $\check{\text{pr}}^{-1}(o)$ (for the natural projection $\check{\text{pr}} : \check{X} \rightarrow B$) is linear with endvertices meeting the proper transforms \check{B}_1 and \check{B}_2 of $B \times \{\infty\}$ and $B \times \{0\}$ respectively. By assumption μ induces a regular complete vector field $\check{\mu}$ on \check{X} tangent to the fiber of $\check{\text{pr}}$ and such that its restriction to $\check{\text{pr}}^{-1}(o)$ vanishes only at double points of the curve $\check{B}_1 \cup \check{\text{pr}}^{-1}(o) \cup \check{B}_2$.

Note that σ is a monoidal transformation with center also at one of these points. Hence $\check{\mu}$ generates a field $\tilde{\mu}$ tangent to the fibers of pr which is complete by construction. Its phase flow preserves the curve $\tilde{B}_1 \cup \text{pr}^{-1}(o) \cup \tilde{B}_2$ and in particular it is identical on the set S of the double points of this curve, i.e. $\tilde{\mu}$ vanishes at these points. For any component F of $\text{pr}^{-1}(o)$ on which $\tilde{\mu}$ is not identically zero, $\tilde{\mu}$ does not vanish on $F \setminus S \simeq \mathbb{C}^*$ since no rational curve but \mathbb{C} or \mathbb{C}^* can be an integral curve of a complete vector field. Therefore, it remains to show that $\tilde{\mu}$ has only isolated zeros on $\text{pr}^{-1}(o)$.

⁷That is, a component that survives the Remmert reduction.

By induction one can suppose that in some local coordinate system (z, w) near a double point of $\check{B}_1 \cup \check{\text{pr}}^{-1}(o) \cup \check{B}_2$ the local equation of $\check{B}_1 \cup \check{\text{pr}}^{-1}(o) \cup \check{B}_2$ is $zw = 0$ and the field $\check{\mu}$ coincides with

$$nz \frac{\partial}{\partial z} - mw \frac{\partial}{\partial w}$$

for natural n and m with $n + m \geq 1$. Furthermore, for a local coordinate system (ξ, η) on \tilde{X} the map σ is given by $(z, w) = (\xi, \xi\eta)$. The direct computation shows that the local form of $\tilde{\mu}$ is

$$n\xi \frac{\partial}{\partial \xi} - (n + m)\eta \frac{\partial}{\partial \eta}$$

which implies the desired conclusion. \square

Remark 7.2. Note that μ is semi-simple, i.e. its phase flow is an algebraic \mathbb{C}^* -action on $B \times \mathbb{P}^1$. Hence the phase flow of $\tilde{\mu}$ is also an algebraic \mathbb{C}^* -action on \tilde{X} and $\tilde{\mu}$ is semi-simple as well.

Notation 7.3. Let ν be a complete vector field on a smooth semi-affine surface X with a rational first integral. Blowing X up at the points of indeterminacy (note that ν vanishes at such points) we can suppose that this integral is a regular morphism $f : X \rightarrow B$ where B is a complete curve. In particular f is either \mathbb{C} - or \mathbb{C}^* -fibration. Let $\hat{f} : \hat{X} \rightarrow B$ be an extension of f to a \mathbb{P}^1 -fibration on a smooth completion \hat{X} of X by an SNC-divisor \hat{D} which is assumed to be pseudo-minimal. Similarly we suppose that the union \hat{E} of complete curves in X does not contain (-1) -curves tangent to ν (and we call such \hat{E} pseudo-minimal). The extension of ν to \hat{X} will be denoted by $\hat{\nu}$ (i.e. $\hat{\nu}$ may have poles). We note that unless f is a twisted \mathbb{C}^* -fibration the set of zeros of $\hat{\nu}$ contains

- (a) either only one section B_0 of \hat{f} (i.e a general integral curve of $\hat{\nu}$ is isomorphic to \mathbb{C}) or
- (b) two sections B_1 and B_2 of \hat{f} (i.e a general integral curve of $\hat{\nu}$ is isomorphic to \mathbb{C}^*)

where the second option is automatic for the untwisted \mathbb{C}^* -fibration. Furthermore, B_0 (resp. at least B_1) must be contained in \hat{D} since otherwise X is not semi-affine.

Lemma 7.4. *Let Notation 7.3 hold and f and either (a) or (b) hold. Suppose that $\mu = \partial/\partial x$ (resp. $\mu = x\partial/\partial x$) is the vector field on $B \times \mathbb{P}_x^1$ in case (a) (resp. (b)). Then there exists a rational birational map $\varphi : \hat{X} \dashrightarrow B \times \mathbb{P}^1$ over B (in particular the restriction of φ over some open Zariski dense subset B^* of B is an isomorphism) for which*

- (1) μ induces a rational vector field $\hat{\mu}$ on \hat{X} such that for some rational function p on B one has $\hat{\nu} = \hat{f}^*(p)\hat{\mu}$;
- (2) in case (b) the restriction of $\hat{\mu}$ to any fiber of \hat{f} has a finite number of zeros;
- (3) if in case (b) $\hat{\mu}|_X$ is regular then $\hat{\mu}$ is semi-simple and the dual graph of the curve $B_1 \cup \hat{f}^{-1}(b) \cup B_2$ is linear for every $b \in f(X)$.

Proof. By Theorem 6.2 there is a birational morphism $\tau : \hat{X} \rightarrow Y$ onto a ruled surface Y over B . Recall that Y is a locally trivial \mathbb{P}^1 -fibration over B [18, Prop. V.2.2],

i.e. there is a Zariski dense open subset B^* of B such that the preimage Y^* of B^* under the projection $Y \rightarrow B$ is naturally isomorphic to $B^* \times \mathbb{P}^1$ (more precisely, such B^* can be chosen as a neighborhood of any b_0 for which the fiber $f^{-1}(b_0)$ is not a singular one). Furthermore, reducing B^* one can suppose that the proper transforms of B_1 and B_2 in Y^* (in case (b)) are disjoint and the restriction of τ to $\hat{X}^* = \tau^{-1}(Y^*)$ yields an isomorphism $X^* \simeq Y^*$. Using the freedom of choice of this isomorphism $Y^* \simeq B^* \times \mathbb{P}^1$ one can suppose now that these proper transforms are $B^* \times \{\infty\}$ and $B^* \times \{0\}$ respectively (and in case (a) the proper transform of B_0 is $B^* \times \{\infty\}$). Extending isomorphism $Y^* \simeq B^* \times \mathbb{P}^1$ we get $\varphi : \hat{X} \dashrightarrow B \times \mathbb{P}^1$. Let us show that φ is the desired rational map.

By the Zariski theorem there is a surface W that dominates both \hat{X} and $B \times \mathbb{P}^1$ over B . In particular in case (b) the proper transforms B'_1 and B'_2 of B_1 and B_2 in W are disjoint (since $B \times \{\infty\}$ and $B \times \{0\}$ are). By Lemma 6.3(2) the dual graph of every singular fiber of the natural morphism $\kappa : W \rightarrow B$ is contractible to the minimal linear subgraph joining two vertices meeting B'_1 and B'_2 respectively. Making all such contractions one gets a morphism $W \rightarrow \tilde{X}$. It remains to note that the morphism $W \rightarrow B \times \mathbb{P}^1$ must factor through $W \rightarrow \tilde{X}$ since one wants to keep the proper transforms of B'_1 and B'_2 disjoint. That is, $\varphi = \psi \circ \chi$ where $\psi : \tilde{X} \rightarrow B \times \mathbb{P}^1$ is a birational morphism over B and $\chi : \hat{X} \dashrightarrow \tilde{X}$ is the birational map (that factors through W).

The image of $\hat{\nu}$ under isomorphism $\hat{X}^* \simeq Y^*$ yields a complete vector field ν_0 on $Y^* \setminus B^* \times \{\infty\} \simeq B^* \times \mathbb{C}_x$ tangent to the fibers of the natural projection onto B^* . Hence in case (a) the restriction of the field ν_0 to every fiber must be proportional to $\mu = \partial/\partial x$ and thus there is a regular function p on B^* for which $\nu_0 = p\mu$. The extension p to B is the desired rational function. In case (b) the argument about existence of p is similar but with $\mu = x\partial/\partial x$ which yields (1).

By Proposition 7.1 and Remark 7.2 in case (b) μ induces a semi-simple field $\tilde{\mu}$ on \tilde{X} whose restriction on any fiber $\text{pr}^{-1}(b)$ of the natural projection $\text{pr} : \tilde{X} \rightarrow B$ has zeros only at the set Z_b of double points of the curve $\tilde{B}_1 \cup \text{pr}^{-1}(b) \cup \tilde{B}_2$ where \tilde{B}_i is the proper transform of B_i . By construction the image I of the exceptional divisor F of the morphism $W \rightarrow \tilde{X}$ does not meet Z_b for any $b \in B$. That is, $\tilde{\mu}$ does not vanish at any point of I and therefore $\tilde{\mu}$ induces a rational vector field on W with poles on F . In particular, the restriction of this rational vector field to any fiber of $\kappa : W \rightarrow B$ has a finite number of zeros which yields (2).

Assume that for some $b \in f(X)$ the fiber $\kappa^{-1}(b)$ has a nonlinear dual graph, or, equivalently, $F \cap \kappa^{-1}(b)$ is not empty. The proper transform of $F \cap \kappa^{-1}(b)$ in \hat{X} is not contained in \hat{D} because of pseudo-minimality assumption. That is, there is a component C in $F \cap \kappa^{-1}(b)$ whose proper transform \hat{C} in \hat{X} meets X . Hence $\hat{\mu}|_X$ is not regular because of poles on \hat{C} . Thus for regularity one needs χ to be an isomorphism which yields (3). \square

We need to consider two essentially different cases: when $f : X \rightarrow B$ is surjective and when it is not. As the next claim shows the assumption that there is a vector field ν tangent to fibers of a \mathbb{C} - and \mathbb{C}^* -fibration in Notation 7.3 is superfluous in the non-surjective case.

Proposition 7.5. *Let $f : X \rightarrow B$ be a non-surjective morphism from a smooth semi-affine surface X into B with general fibers isomorphic to \mathbb{C} or to \mathbb{C}^* . Then there is a complete algebraic vector field tangent to the fibers of f . Furthermore, one can choose this field so that it does not vanish on a given general fiber $E = f^{-1}(b_0)$ of f .*

Proof. Suppose first that f is an untwisted \mathbb{C}^* -fibration (resp. \mathbb{C} -fibration). As we showed in the proof of Lemma 7.4 for every regular value $b_0 \in B$ of f there is a Zariski neighborhood $B^* \subset f(X) \subset B$ for which $Z = f^{-1}(B^*)$ is naturally isomorphic to $B^* \times \mathbb{C}^*$ (resp. $B^* \times \mathbb{C}$) over B^* . Vector field $\hat{\mu}$ from Lemma 7.4 may have poles on $X \setminus Z$. However since $f(X)$ is affine one can choose a regular function h on $f(X)$ with prescribed orders of zeros at points of $f(X) \setminus B^*$ so that $h\hat{\mu}$ yields a regular complete algebraic vector field on X tangent to the fibers of f . Choosing h so that $h(b_0) \neq 0$ we get the second statement.

In the twisted case consider a proper extension $\hat{f} : \hat{X} \rightarrow B$ of f to an SNC-completion $\hat{X} = X \cup \hat{D}$ of X . Then \hat{D} has the only one irreducible component B_0 on which the restriction of \hat{f} is not a constant. Furthermore, $p := \hat{f}|_{B_0}$ makes B_0 a ramified double cover of B . Hence one has a \mathbb{Z}_2 -action α on B_0 for which $B \simeq B_0/\alpha$. Let $X_0 = X \times_B B_0$ and $q : X_0 \rightarrow B_0$, $r : X_0 \rightarrow X$ be the induced morphisms. Since r makes X_0 a ramified double cover of X we have again a \mathbb{Z}_2 -action β on X_0 for which $X = X_0/\beta$. Consider an α -invariant Zariski dense open subset $B_0^* \subset p^{-1}(f(X)) \subset B_0$ for which $q^{-1}(B_0^*)$ is naturally isomorphic to $B_0^* \times \mathbb{C}^*$ where the second factor is equipped with a coordinate x . Then the restriction of β to $q^{-1}(B_0^*)$ is given by $\beta(b, x) = (\alpha(b), g(b, x))$ where $(b, x) \in B_0^* \times \mathbb{C}^*$ and for a fixed b the function $g(b, x)$ is a coordinate on \mathbb{C}^* . That is, $g(b, x)$ coincides either with $e(b)x$ or with $e(b)/x$ with $e(b)$ being a nonvanishing regular function on B_0^* . However the first possibility must be disregarded because we deal with a twisted case.

In order to construct a complete algebraic vector field on X tangent to the fibers of f on X it suffices to construct a complete algebraic vector field on $(p \circ q)^{-1}(f(X))$ tangent to the fibers of q and such that its restriction to $q^{-1}(B_0^*)$ is β -invariant. Note that the field $x \frac{\partial}{\partial x}$ is mapped to $-x \frac{\partial}{\partial x}$ under automorphism $x \rightarrow e(b)/x$ of \mathbb{C}^* . Hence for every function regular function h on B_0^* that is α -antisymmetric the field $hx \frac{\partial}{\partial x}$ is invariant with respect to the β -action. Choosing this function h on B_0 so that its extension to the affine curve $p^{-1}(f(X))$ has zeros of sufficiently high order at points of $p^{-1}(f(X)) \setminus B_0^*$ we guarantee the regular extension of this field to $(p \circ q)^{-1}(f(X))$ which yields the first statement. For the second statement choose B_0^* so that it contains $p^{-1}(b_0)$ and require that h does not vanish on this set. Hence we are done. \square

In particular the presence of \mathbb{C} - and \mathbb{C}^* -fibrations in Proposition 4.1 and Theorem 5.10 implies the following.

Corollary 7.6. *Every normal semi-affine surface X with a complete algebraic vector field on and it without a rational first integral has automatically an open orbit.*

8. RATIONAL FIRST INTEGRAL, II

Lemma 8.1. *Let $\hat{\nu}$ and $\hat{f} : \hat{X} \rightarrow B$ be as in Notation 7.3 and $f : X \rightarrow B$ be surjective. Then*

- (i) *f is an untwisted \mathbb{C}^* -fibration;*

- (ii) every singular fiber of \hat{f} has a linear dual graph $C_1 + \dots + C_n + C + C'_{n'} + \dots + C'_1$ where $n, n' \geq 1$, C is the only (-1) -vertex of this fiber (i.e. $C_i \cdot C_i$ and $C'_i \cdot C'_i \leq -2$), $C_i \subset \hat{D}$ and $C'_i \subset \hat{E}$ while C meets both X and \hat{D} ;
- (iii) the field $\hat{\nu}$ (and therefore ν) is semi-simple.

Proof. Assume that contrary to (i) we deal with case (a) from Notation 7.3. Let $\varphi : \hat{X} \dashrightarrow B \times \mathbb{P}^1$, $\hat{\mu}$ and p be as in Lemma 7.4, and let B^* and $\hat{X}^* \simeq B^* \times \mathbb{P}^1$ be as in the proof of that Lemma. In particular p is regular on B^* and $\hat{\nu} = \hat{f}^*(p)\hat{\mu}$ is locally nilpotent on $\hat{X}^* \setminus B_0 \simeq B^* \times \mathbb{C}_x$ (since $\hat{\nu} = p \frac{\partial}{\partial x}$). That is, the phase flow of this complete vector field $\hat{\nu}$ induces an algebraic \mathbb{C}_+ -action on $\hat{X}^* \setminus B_0$ and therefore on $X = \hat{X} \setminus \hat{D}$. However the algebraic quotient X/\mathbb{C}^+ must be affine (since X is a semi-affine surface) while the surjectivity of $f|_X$ implies that this quotient is B . Hence since B is complete we have to disregard case (a).

Suppose now that we are in case (b) and use notation from Lemma 7.4. By surjectivity of $f|_X$ every singular fiber $\hat{f}^{-1}(b)$ contains an irreducible component C that meets \hat{D} but is not contained in \hat{D} . By Lemma 7.4 $\hat{\mu}$ does not vanish identically on C . Hence p has no pole in b since the field $\hat{\nu} = \hat{f}^*(p)\hat{\mu}|_X$ is regular. The absence of poles on complete curve B means that p is constant and thus we can suppose that $\hat{\mu} = \hat{\nu}$.

This implies in particular that $\hat{\mu}$ is regular on $\hat{X} \setminus \hat{D}$. By Lemma 7.4(3) the fibers of \hat{f} are linear chains of rational curves. Note also that $\hat{D} \cup \hat{E}$ is the union of B_1, B_2 , and all irreducible components of singular fibers with exception of components similar to C . Therefore B_1 and B_2 belong to different connected components of $\hat{D} \cup \hat{E}$. Recall that $B_1 \subset \hat{D}$. Since \hat{D} is connected by the Lefschetz theorem, we see that the connected component containing B_1 (resp. B_2) coincides with \hat{D} (resp. \hat{E}).

Assume that $n = 0$, i.e. C meets section B_1 which implies that C is irreducible in the fiber $\hat{f}^*(b)$. By Lemma 6.3 and pseudo-minimality of \hat{E} one has $n' = 0$ contrary to the fact that $\hat{f}^{-1}(b)$ is singular. Thus $n \geq 1$ and similarly $n' \geq 1$ which yields (ii).

It remains to exclude the case of twisted \mathbb{C}^* -fibration, i.e the case when \hat{D} contains a double section B_0 . Let us replace $\hat{f} : \hat{X} \rightarrow B$ by the natural morphism $\hat{X} \times_B B_0 \rightarrow B_0$ and also replace $\hat{\nu}$, X , and \hat{D} by their lifts to $\hat{X} \times_B B_0$. Then two sections of the modified morphism $\hat{f} : \hat{X} \rightarrow B$ are contained in \hat{D} contrary to the argument before, i.e. we have (i) and (ii).

Lemma 7.4 (3) and the equality $\hat{\nu} = \hat{\mu}$ imply (iii) which concludes the proof. \square

Remark 8.2. (1) It follows from the proof that \hat{E} is connected. Thus the Remmert reduction X_0 of X has only one singularity which is automatically a fixed point of an elliptic \mathbb{C}^* -action associated with ν . Hence Lemma 8.1 can be also obtained from the description of normal \mathbb{C}^* -singularities according to [30, 29]. It can be extracted as well from the DPD-presentation for \mathbb{C}^* -surfaces due to Flenner and Zaidenberg [11].

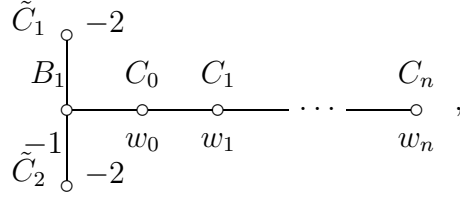
(2) Furthermore, if X_0 is smooth then the Luna slice theorem implies that $X_0 \simeq \mathbb{C}^2$ [24].

(3) In the case when X_0 is not smooth consider the field ν_0 induced by ν on X_0 and the rational first integral f_0 induced by f . One can see that the surjectivity of $f : X \rightarrow B$ is equivalent to the fact that f_0 is not regular on X_0 .

Notation 8.3. Suppose that Notation 7.3 holds and $g : X \rightarrow B'$ is another \mathbb{C} - or \mathbb{C}^* -fibration on X over a complete curve B' such that a general fiber of g is not contained in a general fiber of f and vice versa. Let $\hat{g} : \hat{X}' \rightarrow B'$ be a proper extension of g to an SNC-completion of X by a pseudo-minimal divisor \hat{D}' .

Proposition 8.4. *Let Notation 8.3 hold. Then B_1 (and therefore B_2) is a rational curve and X is a rational surface. Furthermore, if $f : X \rightarrow B$ is surjective then either*

- (1) *the dual graph of \hat{D} (and therefore of \hat{E}) is linear, or*
- (2) *the dual graph of \hat{D} is of form*



where $w_i \leq -2$ for $i \geq 0$. In particular, in (2) \hat{f} has three singular fibers $\hat{f}^{-1}(b_0)$, $\hat{f}^{-1}(b_1)$, and $\hat{f}^{-1}(b_2)$ containing C_0 , \tilde{C}_1 , and \tilde{C}_2 respectively.

Proof. By assumption the restriction of $f|_F : F \rightarrow B$ to a general fiber F of g is not constant. Hence B is rational since F is rational. This implies also that X is rational because f is a \mathbb{C} - or \mathbb{C}^* -fibration.

Assume that f is surjective and \hat{f} has three or more singular fibers or equivalently that the dual graph of \hat{D} is not linear. By Lemma 8.1 B_1 is the only branch point of this graph and the weight of any other vertex is at most -2 . The identical automorphism of X extends to a rational map $\hat{X} \dashrightarrow \hat{X}'$, i.e. there is a reconstruction of the dual graph Γ of \hat{D} into a dual graph Γ' of \hat{D}' .

By pseudo-minimality the weights of linear vertices of Γ' are at most -2 . Hence Corollary 2.5 implies that if Γ' is minimal then it coincides with Γ and otherwise it is obtained from Γ by a sequence of inner and outer blowing up. In particular, Γ' cannot contain a linear 0-vertex and if it has a branch vertex of weight -1 then it is a proper transform of B_1 .

However if $\hat{g} : X \rightarrow B'$ is not surjective then by Proposition 6.6 there must be either a subgraph of type Γ_* from Proposition 6.6 (3b) or a 0-vertex in Γ' . In combination with the previous argument Proposition 6.6 leads to the graph in (2).

Thus it remains to consider the case when $g : X' \rightarrow B'$ is surjective. Let us derive a contradiction by showing that the fibrations \hat{f} and \hat{g} must coincide in this case. As before we see that Γ' is obtained from Γ by a sequence of blowing-ups. Hence by Corollary 2.5 of \hat{D} we can suppose that $\hat{X} = \hat{X}'$.

Let F (resp. G) be a general fiber of \hat{f} (resp. \hat{g}). It suffices to show that F is equivalent to G in $H^2(\hat{X}, \mathbb{Z})$ (because the linear systems $|F|$ and $|G|$ induce \hat{f} and \hat{g} respectively). Let S (resp. S') be the elements of this cohomology group corresponding to the vertices of the dual graphs of $\hat{E} \setminus B_2$ (resp. $\hat{D} \setminus B_1$). Since \hat{X} is contractible to a surface ruled over B we see that the elements of S and S' together with F and B_2 form a basis of $H^2(\hat{X}, \mathbb{Z})$. In particular in this cohomology group $G = kF + lB_2 + M + M'$

where M (resp. M') is an integer linear combination of elements from S (resp. S'). Note that the restriction of the intersection form to S (resp. S') is negative definite by the Zariski lemma [1, page 90]. By the same reason $F \cdot C_i = F \cdot C'_j = G \cdot C_i = G \cdot C'_j = 0$. Hence $(M')^2 = G \cdot M' = 0$ which implies that M' is the zero divisor. Since $G \cdot B_1 = 1$ we have $k = 1$ and $G = F + lB_2 + M$. Then equalities $B_2 \cdot G = 1, G \cdot M = 0$ and $(G)^2 = 0$ imply that

$$l(B_2)^2 + B_2 \cdot M = 0, \quad M^2 + lB_2 \cdot M = 0, \quad \text{and} \quad M^2 + l^2(B_2)^2 + 2lB_2 \cdot M + 2l = 0.$$

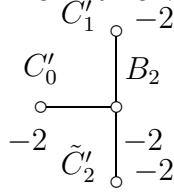
In turn the last three equalities yield $l = 0$ and therefore $M^2 = 0$. The semi-negativity implies that $M = 0$ and $G = F$ which is the desired conclusion. \square

Proposition 8.5. *Let Notation 8.3 hold and $f : X \rightarrow B$ be surjective. Suppose also that \hat{D} has a dual graph different from the graph in Proposition 8.4 (2). Then the Remmert reduction X_0 of X is a normal toric surface.*

Proof. Since \hat{D} is the boundary divisor of a semi-affine surface the Nakai-Moishezon criterion implies that $B_1^2 \geq -1$ and furthermore if $B_1^2 = -1$ then \hat{D} can be contracted to a curve which contains an irreducible component of zero weight and some other components of negative weight. If $B_1^2 > 0$ then after several inner blowing-ups we can make its weight equal to zero. In both cases (and also in the case of $B_1^2 = 0$) one of the end vertices of the dual graph of the resulting curve \tilde{D} is of negative weight and it corresponds to the end-vertex C_n of \hat{D} which is automatically contained in some singular fiber $\hat{f}^{-1}(b)$. Let $C_1 + \dots + C_n + C + C'_n + \dots + C'_1$ be the dual graph of $\hat{f}^{-1}(b)$ as in Lemma 8.1. Then the dual graph of \tilde{D} is of form $F_1 + F_2 + \dots + F_k + C_n$ and furthermore, using elementary transformations from Proposition 2.3 we can suppose that F_1 is of zero weight. In particular, F_1 generates a \mathbb{P}^1 -fibration $\tilde{f} : \tilde{X} \rightarrow \mathbb{P}^1$ on the resulting surface \tilde{X} . Note that by construction F_2 (which may be equal to C_n) is a section of this fibration and the only singular fiber $\tilde{f}^{-1}(a)$ of \tilde{f} is the union G of \hat{E} and the components F_3, \dots, F_k, C_n , and C . Indeed G is connected and does not meet F_1 (i.e. it is contained in a fiber of \tilde{f}), and there is no complete irreducible curve different from a component of G that does not meet both F_1 and F_2 (which would be the case if G is not the whole fiber or in the case of another singular fiber). Furthermore, since $X = \hat{X} \setminus \hat{D} = \tilde{X} \setminus \tilde{D}$ and the dual graph of \hat{D} (or \tilde{D}) is linear the Remmert reduction X_0 of X is a normal Gizatullin surface and in terminology of [9] the curve $\tilde{D} \cup \tilde{f}^{-1}(a)$ is called the extended boundary divisor. A normal Gizatullin surface (say X_0) is toric if and only if the dual graph of the extended boundary divisor for its minimal resolutions of singularities (say X) is linear [9, Lemma 2.20]. This is exactly the case of the dual graph of $\tilde{D} \cup \tilde{f}^{-1}(a)$ and we are done. \square

Proposition 8.6. *Let Notation 8.3 hold and $f : X \rightarrow B$ be surjective. Suppose also that the dual graph of \hat{D} is as in Proposition 8.4 (2). Then*

- (1) X is a surface with an open orbit;
- (2) the phase flow of ν preserves the fibers of g ;
- (3) the Remmert reduction X_0 of X is not generalized Gizatullin unless $n = 0$ and $w_0 = -2$ in which case the dual graph of the curve \hat{E} has the following form



whence the dual graph of every singular fiber $\hat{f}^{-1}(b_i)$ coincides with $[-2, -1, -2]$ for $i = 0, 1, 2$. Furthermore such surface X is unique up to an isomorphism.

Proof. By Proposition 7.5 there is a complete algebraic vector field μ tangent to the fibers of g . Hence ν is not proportional to μ which yields (1).

By Lemma 8.1 $\hat{\nu}$ (resp. ν) is a semi-simple field on \hat{X} (resp. X). In particular the phase flow Φ_t of $\hat{\nu}$ preserves the curve $F_\infty = \tilde{C}_1 \cup B_1 \cup \tilde{C}_2$ which is the fiber of \hat{g} . Thus for any time parameter t and every other fiber F of \hat{g} the image $\Phi_t(F)$ does not meet F_∞ which implies that \hat{g} is constant on $\Phi_t(F)$, i.e. it is again a fiber of \hat{g} . This yields (2).

By Lemma 8.1 the dual graph of $\hat{f}^{-1}(b_1)$ is

$$\tilde{C}_1 + \tilde{C} + \tilde{C}_{n'}'' + \dots + \tilde{C}_1''$$

where \tilde{C} is the only (-1) -vertex, any other weight is at most -2 , and $\tilde{C}_1^2 = -2$. Since this fiber of a \mathbb{P}^1 -fibration must be contractible to a 0-vertex, contracting \tilde{C} and \tilde{C}_1 consequently we see that $n' = 1$ and $(C_1'')^2 = -2$. The same is valid for the fiber $\hat{f}^{-1}(b_2)$, and in the case of $n = 0$ and $w_0 = -2$ it is also true for $\hat{f}^{-1}(b_0)$.

Assume that $w_0 \leq -3$. Consider the dual graph

$$C_0 + C_1 + \dots + C_n + C + C_{n'}' + \dots + C_1'$$

of $\hat{f}^{-1}(b_0)$. Then the connected curve

$$G = C_1 + \dots + C_n + C + C_{n'}' + \dots + C_1' + B_2 + \tilde{C}_1' + \tilde{C}_2'$$

is a fiber of \hat{g} since it contains all complete curves in \hat{X} that do not meet the fiber F_∞ or the section C_0 . Note that B_2 is branch point of the dual graph of G and all branches but

$$\mathcal{B} = C_1 + \dots + C_n + C + C_{n'}' + \dots + C_1'$$

are non-contractible (because we know already that the weights of \tilde{C}_1' and \tilde{C}_2' are -2). Thus the latter must be contractible since any fiber is contractible to a 0-vertex and furthermore, after this contraction the weight of B_2 must become -1 in the graph $[-2, -1, -2]$. Hence as in the proof of Lemma 6.6 we conclude that B_2 is a multiple component of \hat{g} and therefore each component of \mathcal{B} is also multiple. In particular, C is a multiple component of g , i.e. $C \cap X$ is a singular fiber of g .

Suppose that η is a complete algebraic vector field without a rational first integral. By Proposition 4.1 and Theorem 5.10 there must a \mathbb{C}^* - or \mathbb{C} -fibration over \mathbb{C} such that its fibers are transformed into each other by the phase flow of η and g is the only candidate for this fibration (since its minimal extension \hat{g} to a \mathbb{P}^1 -fibration is determined uniquely by the form of \hat{D}). However by Remark 5.11 the curve $C \cap X$ is preserved by this flow.

Similarly, if $C \cap X$ is a component of a fiber of a rational first integral of some complete vector field it is preserved by the phase flow of this field (in particular this is

true for ν and μ). If we have $w_0 \leq -3$ or $n \geq 1$ then g is the unique \mathbb{C}^* - or \mathbb{C} -fibration on X thus the curve $C \cap X$ is preserved by all complete vector fields. Since the image of $C \cap X$ in X_0 is a curve X_0 cannot be generalized Gizatullin when $w_0 \leq -3$ or $n \geq 1$.

Thus we have the desired form of the dual graph of \hat{E} except for the fact that the weight k of B_2 is still unknown. Note first that $k \leq -2$ since otherwise the intersection matrix of \hat{E} is not negative definite contrary to the Grauert criterion of contractibility [1, Theorem III.2.1]. Then contraction of fibers $\hat{f}^{-1}(b_0)$, $\hat{f}^{-1}(b_1)$, and $\hat{f}^{-1}(b_2)$ to 0-vertices transforms \hat{X} into a Hirzebruch surface \hat{X}' with proper transforms B'_1 and B'_2 of B_1 and B_2 as disjoint sections. Their weights k_1 and k_2 depends on the choice of contraction of these three fibers but in any case $k_1 + k_2 = -1 + k + 3 = k + 2 \leq 0$. Since these sections are disjoint it follows from [17, page 518] that one of them is of negative weight (say $-m$) and the weight of the other is at least m which yields $k = -2$ and we are done with the form of the graph of E .

Furthermore, we see now that \hat{X}' can be chosen as $\mathbb{P}^1 \times \mathbb{P}^1$ and \hat{X} is obtained from \hat{X}' by some standard blowing up in three fibers of a natural morphism $\hat{X}' \rightarrow \mathbb{P}^1$. Since the group of automorphisms of \mathbb{P}^1 acts transitively on the triples of distinct points we get the claim about uniqueness. \square

Example 8.7. Let X_0 be from Proposition 8.6. Then its singularity (whose minimal resolution is given by the graph of \hat{E}) is of so-called type $-D_4$ where a singularity of type $-D_{n+1}$ is locally isomorphic to the hypersurface $yx^2 + y^n + z^2 = 0$ in $\mathbb{C}_{x,y,z}^3$. In fact one can see that X_0 is globally isomorphic to the hypersurface $y(x^2 + y^2) + z^2 = 0$. The elliptic \mathbb{C}^* -action on it is given by $(x, y, z) \rightarrow (\lambda^2 x, \lambda^2 y, \lambda^3 z)$ for $\lambda \in \mathbb{C}^*$ and it corresponds to a semi-simple field σ . Three \mathbb{C}^* -fibrations associated with the three different strings $[-2, -1, -2]$ in the graph of \hat{D} are given by the functions y , $x + \sqrt{-1}y$, and $x - \sqrt{-1}y$. By Proposition 7.5 there are complete algebraic vector fields $\sigma_1, \sigma_2, \sigma_3$ on the hypersurface tangent the fibers of these functions respectively. Note that the only curve tangent to both σ and σ_1 is given by $y = 0$, i.e this curve is invariant under their phase flows. However it is not invariant under the phase flows of σ_2 or σ_3 and hence one can check that the natural action of $\text{AAut}_{\text{hol}}(X_0)$ has the smooth part of X_0 as an open orbit. In particular X_0 is generalized Gizatullin (another proof of this fact will be considered in the last section) and hence it is a surface described in Proposition 8.6 (3). For $n \geq 4$ there is only one regular \mathbb{C}^* -fibration on the corresponding surface (given by function y) which is therefore only a surface with an open orbit.

Theorem 8.8. *Let X_0 be a normal generalized Gizatullin surface such that for a complete algebraic vector field ν_0 on X_0 there is a surjective rational first integral $f_0 : X_0 \dashrightarrow B$ into a complete curve B . Then*

- (1) *either X_0 is toric (and in particular a Gizatullin surface) or X_0 is the Remmert reduction of X from Proposition 8.6;*
- (2) *up to a constant nonzero factor ν_0 is semi-simple.*

Proof. Statement (2) follows from 8.1(iii). For (1) consider a birational morphism $X \rightarrow X_0$ from a semi-affine surface X such that f_0 induces a surjective morphism $f : X \rightarrow B$ with general fibers isomorphic to \mathbb{C} or \mathbb{C}^* . If a complete algebraic vector field μ_0 on X_0 non-proportional to ν_0 has also a rational first integral we can suppose

that it induces a similar morphism $g : X \rightarrow B'$. Then Propositions 8.5 and 8.6 imply the desired conclusion.

If μ_0 does not have a rational first integral then by Theorems 3.4 and 5.10 there is a morphism $g_0 : X_0 \rightarrow B'$ with general fibers isomorphic to \mathbb{C} or \mathbb{C}^* . Since f_0 has indeterminacy points its general fibers are different from the general fibers of g_0 and therefore the fibration $g : X \rightarrow B'$ (induced by g_0) is different from $f : X \rightarrow B$. By Proposition 7.5 there is a complete vector field tangent to the fibers of g and we are done again by Proposition 8.5. \square

As another consequence of this section we have a generalization of the fact which in the case of \mathbb{C}^2 can be extracted from [5].

Theorem 8.9. *Let X be a normal affine algebraic surface which admits a nonzero complete algebraic vector field. Then either:*

(1) *all complete algebraic fields share the same rational first integral (i.e. there is a rational map $f : X \dashrightarrow B$ such that all complete algebraic vector fields on X are tangent to the fibers of f), or*

(2) *X is rational with an open orbit and, furthermore, for every complete algebraic vector field ν on X there is a regular function $f : X \rightarrow \mathbb{C}$ (depending on ν) with general fibers isomorphic to \mathbb{C} or \mathbb{C}^* such that the flow of ν sends fibers of f to fibers of f .*

Proof. Assume that X has no open orbit, i.e. ν must have a rational first integral $f : X \dashrightarrow B$ by Proposition 7.5. Then we have (1).

For (2) consider a surface X with an open orbit. Because of Proposition 7.5 we can suppose again that ν has a rational first integral. If such an integral is surjective then by Theorem 8.8 we may deal either with a toric surface or with the surface from Proposition 8.6. In the latter case the second statement of Proposition 8.6 yields the desired regular function $f : X \rightarrow \mathbb{C}$. In the former case the existence of such f was established in [22] or it can be extracted from explicit description of algebraic \mathbb{C}^* -actions on Gizatullin surfaces in [9]. Now suppose that every complete algebraic vector field on X has a regular rational first integral. Then such integrals for non-proportional fields lead to different \mathbb{P}^1 -fibrations on a completion of X satisfying the assumption of Proposition 8.4 whence X is rational. Hence for any \mathbb{C} - or \mathbb{C}^* -fibration $X \rightarrow B$ one has $B \simeq \mathbb{C}$. In any case this regular rational first integral for ν can be viewed as a desired $f : X \rightarrow \mathbb{C}$ in (2) and we are done. \square

9. PROOF OF THE NECESSITY PART OF THE MAIN THEOREM

Notation 9.1. In this section we suppose that \hat{X} is an SNC-completion of a smooth rational semi-affine surface X . As usual the dual graph of $\hat{D} = \hat{X} \setminus X$ will be denoted by Γ .

Definition 9.2. We say that an irreducible curve $F \subset X$ is distinguished if for any \mathbb{C} - or \mathbb{C}^* -fibration $f : X \rightarrow \mathbb{C}$ this curve is contained in a singular fiber of f .

As we mentioned before in Remark 5.11 the following fact holds.

Lemma 9.3. *Let ν be a complete vector field on X that sends fibers of f to fibers of f . Then the flow of ν preserves every singular fiber of f .*

In combination with Theorem 8.9 this leads to the following important technical tool which has already appeared implicitly in Proposition 8.6.

Proposition 9.4. *If a semi-affine surface X contains a distinguished curve then its Remmert reduction X_0 is not weakly quasi-minimal.*

Proof. Let C be a distinguished curve and let ν be a complete algebraic vector field on X_0 . Then by Theorem 8.9 the flow of ν preserves a \mathbb{C} - or \mathbb{C}^* -fibration f . Since C by definition belongs to a singular fiber of f it is preserved by Remark 5.11. \square

Notation 9.5. Recall that Γ is contractible to a minimal graph Γ_m . The set of branch points of Γ that survive this contraction and remain branch points after it will be denoted by $\text{Br}_m(\Gamma)$. That is, there is a natural bijection between $\text{Br}_m(\Gamma)$ and the set $\text{Br}(\Gamma_m)$ of branch points of Γ_m . Note that $\text{Br}(\Gamma_m)$ (and therefore $\text{Br}_m(\Gamma)$) can be presented as a union $T_{0,m} \cup T_{1,m}$ (resp. $T_0 \cup T_1$) of two disjoint sets where $T_{0,m}$ consists of all branch points E of valency 3 in Γ_m such that

- two of branches at E are just (-2) -vertices;
- by elementary transformations (as in Proposition 2.3) on connected components of $\Gamma_m \ominus \text{Br}(\Gamma_m)$ one can make the weight of E equal to -1 .

That is, the smallest subgraph of Γ_m containing E and its neighbors is of type Γ_* from Lemma 6.6. Consider the connected components of $\Gamma \ominus T_1$. Those of them that are not contractible will be denoted by $\Gamma_0, \Gamma_1, \dots, \Gamma_n$ (note that they are in a natural one-to-one correspondence with connected components of $\Gamma_m \ominus T_{1,m}$). We suppose also that Γ_i corresponds to a curve D_i contained in \hat{D} .

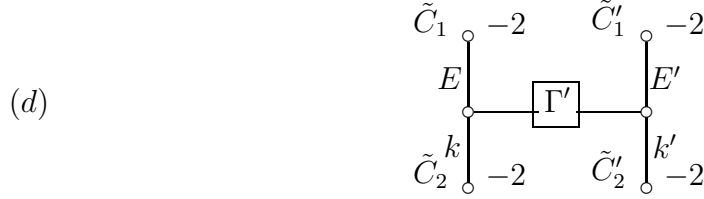
Lemma 9.6. *The notions of T_i and Γ_i are independent of the choice of a minimal graph for Γ .*

Proof. Let Γ be contractible to another minimal graph Γ'_m . Suppose that $\text{Br}(\Gamma'_m)$ and T'_i , $i = 0, 1$ are as in Notation 9.5 with Γ' instead of Γ . Contractions of Γ to Γ_m and Γ'_m yield a reconstruction between the last two minimal graphs. By Proposition 2.4 this leads to a natural bijection $\Phi : \text{Br}(\Gamma_m) \rightarrow \text{Br}(\Gamma'_m)$ and a natural bijection Θ between connected components of $\Gamma_m \ominus \text{Br}(\Gamma_m)$ and $\Gamma'_m \ominus \text{Br}(\Gamma'_m)$. If \mathcal{C} is such a component then Corollary 2.5 implies that \mathcal{C} is a (-2) -vertex iff $\Theta(\mathcal{C})$ is. Note also that Φ preserves the valency of branch points by Lemma 2.4. Hence it maps $T_{0,m}$ onto the similar set $T'_{0,m}$. Therefore there are natural bijections between T_i and T'_i , $i = 0, 1$ and between connected components of $\Gamma_m \ominus T_{1,m}$ and $\Gamma'_m \ominus T'_{1,m}$ which implies the desired conclusion. \square

Connectedness of \hat{D} and the description of vertices from T_0 in Notation 9.5 imply the following.

Lemma 9.7. *Each graph Γ_i has one of the following configurations:*

- (a) *linear graph;*
- (b) *circular graph;*



where Γ' is a linear graph or empty. Furthermore

- in (b) and (d) one has $\Gamma_i = \Gamma$;
- in (a) there are at most two neighbors of Γ_i in Γ , each of them is contained in T_1 and joined by an edge with an end-vertex of Γ_i ;
- in (c) there is at most one neighbor of Γ_i in Γ , it is contained in T_1 and joined by an edge with the right end-vertex of Γ' .

Lemma 9.8. *Let $\hat{f} : \hat{X} \rightarrow \mathbb{P}^1$ be a proper extension to \hat{X} of a \mathbb{C} - or \mathbb{C}^* -fibration $f : X \rightarrow \mathbb{C}$, i.e. some fiber of \hat{f} has a support in \hat{D} . Then this fiber must in fact have this support in some D_i .*

Proof. Making contractions in \hat{D} we get a pseudo-minimal extension $\hat{g} : \hat{X}' \rightarrow \mathbb{P}^1$ of f with dual graph Γ' for $\hat{D}' = \hat{X}' \setminus X$. By Proposition 6.6 Γ' contains either a linear 0-vertex C or a subgraph of type Γ_* which includes a chain $[-2, -1, -2]$ consisting of vertices $C_1 + E + C_2$. Since linear vertices remain linear under further contraction of Γ' to a minimal graph Γ_m (unless they are contracted) we see that the image of C in Γ_m does not belong to $T_{1,m}$. That is, it belongs to a connected component of $\Gamma_m \ominus T_{1,m}$ corresponding to some Γ_i . Hence the preimage of C in D (which is a support of a fiber of \hat{f}) is contained in D_i .

If Γ' contains the chain $C_1 + E + C_2$ then the the images of C_1 and C_2 remain linear in Γ_m while the image of E is either a linear vertex or a branch point of valency 3. In the second case the image of E belongs to $T_{0,m}$. However in both case the image of E is not in $T_{1,m}$, and the image of the whole chain is contained in a connected component of $\Gamma_m \ominus T_{1,m}$ which implies the desired conclusion as before. \square

Proposition 9.9. *Let $\hat{f} : \hat{X} \rightarrow \mathbb{P}^1$ and $\hat{g} : \hat{X} \rightarrow \mathbb{P}^1$ be proper extensions to \hat{X} of \mathbb{C} - or \mathbb{C}^* -fibrations on X . Suppose that there is a fiber F of \hat{f} and a fiber G of \hat{g} with supports in different D_i 's. Then \hat{f} and \hat{g} coincide (up to an automorphism of \mathbb{P}^1) and furthermore every \mathbb{P}^1 -fibration $h : \hat{X} \rightarrow \mathbb{P}^1$ which extends a \mathbb{C} - or \mathbb{C}^* -fibration on X coincides with \hat{f} .*

Proof. By assumption $F \cdot G = 0$. Hence if F and G are not algebraically equivalent they generate a two-dimensional subspace of $H^2(\hat{X})$ such that the restriction of the intersection form to it is non-negative contrary to the Hodge index theorem. Thus this divisors are equivalent and \mathbb{P}^1 -fibrations \hat{f} and \hat{g} (generated by the linear systems of these divisors) coincide (up to an automorphism of the image \mathbb{P}^1). By Lemma 9.8 \hat{h} has a fiber with support in some D_j . Note that D_j does not contain the support of either F or G . Hence \hat{h} coincides with either \hat{f} or \hat{g} and we are done. \square

Corollary 9.10. *Suppose that \hat{X} admits two distinct \mathbb{P}^1 -fibrations $\hat{f} : \hat{X} \rightarrow \mathbb{P}^1$ and $\hat{g} : \hat{X} \rightarrow \mathbb{P}^1$ that extend \mathbb{C} - or \mathbb{C}^* -fibrations on X . Then both of them have fibers with support in the same D_i (say, D_0) and no fibers with support in any D_j where $j \neq i$.*

Proposition 9.11. *If X is generalized Gizatullin then distinct fibrations \hat{f} and \hat{g} as in Corollary 9.10 exist. In particular, Γ_0 cannot be contracted to a 0-vertex or it cannot be in form (c) with empty Γ' .*

Proof. Assume that there is only one \mathbb{P}^1 -fibration \hat{f} of this type. Then its restriction $f : X \rightarrow B := f(X) \subset \mathbb{C}$ has no singular fibers by Proposition 9.4 and Lemma 9.3, i.e. f is a locally trivial \mathbb{C} - or \mathbb{C}^* -fibration over B . Furthermore, B is not hyperbolic since otherwise for every complete vector field ν the image of any integral curve under f must be constant, i.e. ν must be tangent to the fibers of f and X has no open orbit. That is, B is either \mathbb{C} or \mathbb{C}^* . This implies that X is isomorphic to either \mathbb{C}^2 , $\mathbb{C} \times \mathbb{C}^*$, $(\mathbb{C}^*)^2$ or a twisted locally trivial \mathbb{C}^* -fibration over \mathbb{C}^* . In the first three cases our assumption is obviously wrong.

In the last case \hat{D} contains two fibers of \hat{f} whose graphs under pseudo-minimality assumption is the chain $[-2, -1, -2]$ by Lemma 6.6. That is, Γ is of form (4) with Γ' being the component C_0 of \hat{D} that is ramified double cover of \mathbb{P}^1 under \hat{f} and $k = k' = -1$. In order to avoid the existence of another fibration \hat{g} one needs to require that $C_0^2 \leq -1$.

Let us show that this is impossible. Contracting fibers $\tilde{C}_1 + E + \tilde{C}_2$ and $\tilde{C}'_1 + E' + \tilde{C}'_2$ to 0-vertices we obtained a Hirzebruch $\bar{X} \simeq \Sigma_l$ and the induced fibration by $\bar{f} : \bar{X} \rightarrow \mathbb{P}^1$ with the proper transform \bar{C}_0 of C_0 playing the role of a ramified double section of \bar{f} . By construction $\bar{C}_0 \cdot \bar{C}_0 = C_0 \cdot C_0 + 4 \leq 3$. Let S be the negative section of \bar{f} (i.e. $S^2 = -l$) [17, page 518] and F be any fiber of \bar{f} that intersects \bar{C}_0 in two points. At least one of them does not belong to S . Blowing this point up and contracting the proper transform of F we get another Hirzebruch surface Σ_{l-1} (since the proper transform of S has selfintersection $1 - l$). Note that this procedure does not change the selfintersection of the proper transform of \bar{C}_0 . Repeating it we reconstruct \bar{X} into $\mathbb{P}^1 \times \mathbb{P}^1$ containing the proper transform \check{C}_0 of \bar{C}_0 . Hence $\check{C}_0 \cdot \check{C}_0 \leq 3$ while for a double section in $\mathbb{P}^1 \times \mathbb{P}^1$ this number should be at least 4. This contradiction concludes the proof. \square

Example 9.12. The last surface S which is a twisted locally trivial \mathbb{C}^* -fibration over \mathbb{C}^* is rather interesting. Given a coordinate z on a fiber \mathbb{C}^* one can suppose that the monodromy around the puncture in the base is given by $z \rightarrow 1/z$. Treating \mathbb{C}^* as a complexification of a circle both in the base and in the fiber one can see that S is nothing but complexification of the Klein bottle. An SNC-completion of S can be constructed

in the following way. Consider the parabola C given by $x - y^2 = 0$ in $\mathbb{P}_x^1 \times \mathbb{P}_y^1$. Blow up point $(0, 0)$ (resp. (∞, ∞)) and an infinitely near point. The resulting surface \hat{S} is the desired completion of S with fibers over $x = 0$ and $x = \infty$ being $[-2, -1, -2]$ -chains in $\hat{S} \setminus S$ and the proper transform C_0 of C playing the role of the ramified double cover with $C_0^2 = 0$. There are only two \mathbb{C}^* -fibrations x and y^2/x on S corresponding to the \mathbb{P}^1 -fibrations associated with the chains $[-2, -1, -2]$ and vertex C_0 respectively. Both x and y^2/x are rational first integrals (the first one for the complete algebraic field $\nu_1 = (y^2 - x)\partial/\partial y$ and the second one for $\nu_2 = 2x\partial/\partial x + y\partial/\partial y$). There is also a complete algebraic field $\nu = \nu_1 + \nu_2$ ⁸ for which neither x nor y^2/x is a rational first integral (and which therefore has no rational first integral at all). Note that the function x yields the preserved fibration on S . One can conclude from this that the Kodaira dimension of the foliation associated with ν is 1 since it follows from our construction in Section 5 that in the case of Kodaira dimension 0 the adapted fibration cannot be a twisted \mathbb{C}^* -fibration.

Remark 9.13. Proposition 9.11 does not hold in general for surfaces with open orbits. Indeed, consider surfaces $yx^2 + y^n + z^2 = 0$, $n \geq 4$ with singularities of type $-D_n$ mentioned in Example 8.7. Each of them (say S) has a complete algebraic field tangent to the \mathbb{C}^* -fibration described in that example and the semi-simple field associated with the \mathbb{C}^* -action $(x, y, z) \rightarrow (\lambda^{n-1}x, \lambda^2y, \lambda^nz)$. Hence it has an open $\text{AAut}_{\text{hol}}(S)$ -orbit. On the other hand there is an SNC-completion of S with dual graph of the divisor at infinity as in Proposition 8.4 whose form in combination with Lemma 6.6 yields a unique \mathbb{C}^* -fibration and the absence of \mathbb{C} -fibrations on S .

Proposition 9.14. *Let Notations 9.1 and 9.5 hold and the Remmert reduction X_0 of X be a generalized Gizatullin surface. Then T_1 is empty and in particular Γ has one of four configurations from Lemma 9.7.*

Proof. By Lemma 9.7 it suffices to consider configurations (a) and (c) and show that T_1 is empty. Assume that $\Gamma \neq \Gamma_0$, i.e. there is a vertex $V \in T_1$ of valency $m \geq 3$ adjacent to the right end of Γ_0 .

Configuration (a). By Proposition 2.3 we can suppose that the left end E_0 of Γ_0 is a 0-vertex.

That is, \mathbb{P}^1 -fibration $\hat{f} : \hat{X} \rightarrow \mathbb{P}^1$ induced by E_0 is an extension

- (i) of a \mathbb{C} -fibration if E_0 has no neighbor from T_1 ;
- (ii) of a \mathbb{C}^* -fibration if E_0 has such a neighbor W (where we allow equality $W = V$).

In (i) (and even in (ii) when $W \neq V$) V is not a section of the fibration \hat{f} and is contained in a fiber $\hat{f}^{-1}(a)$ of \hat{f} . By Lemma 6.8 in case (i) (resp. (ii)) $\hat{f}^{-1}(a)$ contains at least $m-1 \geq 2$ (resp. $m-2 \geq 1$) irreducible components C_i such that $F_i := C_i \cap X \simeq \mathbb{C}$ and C_i does not meet $D_0 \cup V$. In particular, $f^{-1}(a)$ is singular (being non-irreducible in case (i) or non-isomorphic to \mathbb{C}^* in case (ii)).

Given a different \mathbb{C} - or \mathbb{C}^* -fibration $g : X \rightarrow \mathbb{C}$ by the Zariski theorem we can find an SNC-completion $\bar{X} = X \cup \bar{D}$ of X which dominates \hat{X} and admits a proper extension \bar{g} of g . The preimage \bar{D}_0 of D_0 in \bar{D} has dual graph $\bar{\Gamma}_0$ which contains the dual graph G_0 of the preimage of E_0 and by Proposition 9.9 the dual graph G of the support of a

⁸Completeness of ν can be extracted from completeness of ν_1 and ν_2 and the fact that $[\nu_1, \nu_2] = -\nu_1$.

fiber of \bar{g} . By the Zariski lemma (e.g., see [1, p. 90]) $G_0 \setminus G$ cannot be empty, i.e. there are two neighbors of G . Thus g is a \mathbb{C}^* -fibration on X . Note that C_i is contained in the fiber of \bar{g} because it does not meet $\bar{D}_0 \cup V$. Hence $F_i \simeq \mathbb{C}$ is a component of a singular fiber of g and thus it is a distinguished curve. Since it survives the Remmert reduction we see that X_0 is not generalized Gizatullin by Proposition 9.4. A contradiction, i.e. T_1 must be empty in this case.

When $W = V$ we deal only with \mathbb{C}^* -fibrations. By Proposition 9.11 Γ_0 contains a neighbor E_1 of V to the right of E_0 . Blowing up the edge between E_1 and V , if necessary, we can suppose that Γ_0 contains at least three vertices. Using reconstructions as in Proposition 2.3 we can change Γ_0 so that it contains now a 0-vertex E somewhere in the middle (i.e. E is not a neighbor of V). Lemma 6.8 supplies us as before with a distinguished curve $F \simeq \mathbb{C}$ which concludes consideration of configuration (a).

Configuration (c). Consider $\hat{f} : \hat{X} \rightarrow \mathbb{P}^1$ induced by the chain $\tilde{C}_1 + E + \tilde{C}_2 = [[-2, -1, -2]]$. Note that \hat{f} yields a \mathbb{C}^* -fibration f on X and V is not a section of \hat{f} . Hence we can repeat the previous argument which finished the proof. \square

Now we are able to prove one direction of the Theorem 1.4, namely the necessary condition for a semi-affine surface to be generalized Gizatullin.

Proposition 9.15. *Let X be a semi-affine surface which is generalized Gizatullin, then X possesses a completion \bar{X} where the dual graph of $\bar{X} \setminus X$ is of one of the following forms*

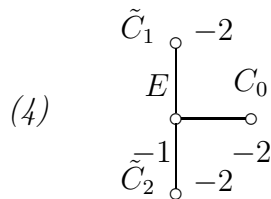
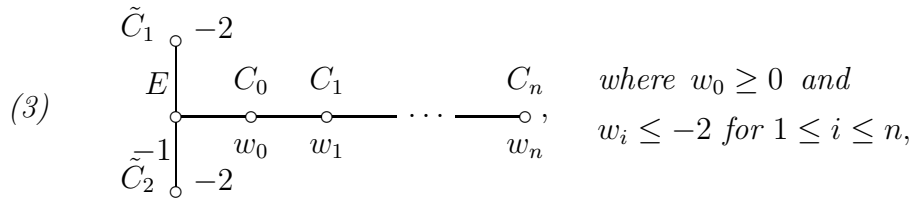
(1) *a standard zigzag or a linear chain of three 0-vertices (i.e. Gizatullin surfaces and $\mathbb{C} \times \mathbb{C}^*$),*

(2) *circular graph with the following possibilities for weights*

(2a) $((0, 0, w_1, \dots, w_n))$ where $n \geq 0$ and $w_i \leq -2$,

(2b) $((0, 0, w))$ with $-1 \leq w \leq 0$ or $((0, 0, 0, w))$ with $w \leq 0$,

(2c) $((0, 0, -1, -1));$



$$(5) \quad \begin{array}{c} \tilde{C}_1 \quad -2 \\ | \\ E \\ | \\ \tilde{C}_2 \quad -1 \\ | \\ \tilde{C}_2 \quad -2 \end{array} \begin{array}{c} C_0 \\ | \\ w_0 \end{array} \begin{array}{c} C_1 \\ | \\ w_1 \end{array} \cdots \begin{array}{c} C_n \\ | \\ w_n \end{array} \begin{array}{c} \tilde{C}'_1 \quad -2 \\ | \\ E' \\ | \\ \tilde{C}'_2 \quad k' \\ | \\ \tilde{C}'_2 \quad -2 \end{array}, \quad \begin{array}{l} \text{where } w_0 \geq 0 \\ \text{and } w_i \leq -2 \text{ for } 1 \leq i \leq n, \\ \text{moreover } k' \leq -1 \text{ if } n = 0 \\ \text{or } k' \leq -2 \text{ if } n > 0, \end{array}$$

$$(6) \quad \begin{array}{c} \tilde{C}_1 \quad -2 \\ | \\ E \\ | \\ \tilde{C}_2 \quad -1 \\ | \\ \tilde{C}_2 \quad -2 \end{array} \begin{array}{c} \tilde{C}'_1 \quad -2 \\ | \\ E' \\ | \\ \tilde{C}'_2 \quad k' \\ | \\ \tilde{C}'_2 \quad -2 \end{array}, \quad \text{for } k' \geq -1.$$

Proof. For the case of a linear dual graph the Proposition is well known. Let us consider first circular graphs. By [8, Proposition 3.28] they can be reduced via birational transformations to the following standard (and “essentially unique”) forms:

- (i) $((0_{2k}, w_1, \dots, w_n))$ with $k \geq 0, n > 0$, and $w_i \leq -2$; or
- (ii) $((0_l, w))$ with $l > 0$ and $w \leq 0$; or
- (iii) $(0_{2k}, -1, -1))$ with $k \geq 0$

where subindex reflects the number of consequent zero weights. Note that in (i) we disregard the case of $k = 0$ (since the intersection matrix of such a graph is negative definite contrary to the Nakai-Moishezon criterion) and $k \geq 2$ (to avoid contradiction with the Hodge index theorem). Similarly, in (ii) we omit the case of $l = 1$ and $w \leq -1$ because otherwise we get only one \mathbb{P}^1 -fibration as in Proposition 9.10 contrary to Proposition 9.11. The Hodge index theorem implies also that in (ii) $l \leq 3$. Hence remaining possibilities in (i) and (ii) produce (2a) and (2b). By the same arguments in (ii) we have to consider only weights as in (2c).

By Proposition 9.14 it suffices now to consider Γ as in configurations (c) and (d) in Lemma 9.7 and show that Γ' is of desired form.

Consider first the case when there exists a nonnegative vertex in a minimal graph of Γ' . After blowing up which keeps the graph linear we can always suppose that it is actually of weight 0. Using operations of of form $[[v, 0, w]] \rightarrow [[v - 1, 0, w + 1]]$ as in Proposition 2.3 we can suppose furthermore that this vertex is the left endpoint C_0 of Γ' and the weight k in configurations (c) and (d) from Lemma 9.7 is -1 . In particular $\tilde{C}_1 + E + \tilde{C}_2$ is the support of a fiber F of \mathbb{P}^1 -fibration $\hat{f} : \hat{X} \rightarrow \mathbb{P}^1$ for which the vertices different from C_0 (being disjoint from F) must be contained in another fiber. In particular they are all of negative weight and making contraction one can suppose that $C_i^2 \leq -2$ for $1 \leq i \leq n$ while C_0 is of nonnegative weight. In the case of $n \geq 1$ the weight of E' cannot be -1 since otherwise the chains $\tilde{C}'_1 + E' + \tilde{C}'_2$ and $\tilde{C}_1 + E + \tilde{C}_2$ are contractible to disjoint 0-curves which must be equivalent by the Hodge index theorem contrary to the fact that one of them meets C_0 while the other does not. This describes configurations (3) and (5) completely.

Now consider the case when there is no nonnegative vertex in a minimal graph of Γ' . If in (d) Γ' is not empty then again the divisor

$$C_1 + \dots + C_n + \tilde{C}'_1 + E' + \tilde{C}'_2$$

is contained in the same fiber of \hat{f} and we can suppose that $(E')^2 \leq -1$ and $C_i^2 \leq -2$. By Corollary 2.5 we see that this minimal graph is unique and \hat{f} is the only extension of a \mathbb{C} - or \mathbb{C}^* -fibration on X contrary to Proposition 9.11. Thus Γ' empty in this case and we have (6) (condition $k' \geq -1$ is necessary to provide a second \mathbb{P}^1 -fibration not equal to \hat{f}).

In the absence of a nonnegative vertex for (c) we have $k = -1$ and unless $w_0 = -2$ there is again the same contradiction with Proposition 9.11. Thus $w_0 = -2$ and we have three \mathbb{P}^1 -fibrations associated with the chains $\tilde{C}_1 + E + \tilde{C}_2$, $\tilde{C}_1 + E + C_0$, and $\tilde{C}_2 + E + C_0$ respectively. However if $n \geq 1$ the restriction of any of the last two fibrations to X is not \mathbb{C}^* -fibration because contrary to Lemma 6.6(3b) the $\tilde{C}_i + E + C_0$ meets not only the ramified double cover but also C_1 . This leads to case (4) and we are done. \square

10. PROOF OF THE SUFFICIENCY PART OF THE MAIN THEOREM

Lemma 10.1. *Let X be a smooth semi-affine surface and $\mathrm{AAut}_{\mathrm{hol}}(X)$ be as in the introduction. Suppose that $f_i : X \rightarrow \mathbb{C}$, $i = 1, 2$ are either \mathbb{C} or \mathbb{C}^* -fibrations such that the intersection of every pair of non-singular fibers of f_1 and f_2 is a finite non-empty set. Let S_i be the union of singular fibers of f_i . Then there is an open orbit U of the natural $\mathrm{AAut}_{\mathrm{hol}}(X)$ -action on X such that its complement is contained in $S_1 \cap S_2$.*

Proof. Let $x \notin S_1 \cap S_2$ and let U be the orbit of x . Say x is contained in a non-singular fiber E_1 of f_1 . By Lemma 7.5 $E_1 \subset U$, i.e. we can suppose that x is an arbitrary point of E_1 . In particular we can suppose that $x \in E_1 \cap E_2$ where E_2 is a given non-singular fiber of f_2 . By Lemma 7.5 $E_2 \subset U$. Similarly any given fiber of f_1 is contained in U and we are done. \square

The following Proposition proves the sufficiency part of Theorem 1.4.

Proposition 10.2. *Every normal affine algebraic surface X_0 with a dual graph appearing in Proposition 9.15 is generalized Gizatullin surface.*

Proof. If X_0 is a Gizatullin surface then it is known that X_0 is already quasi-homogeneous under the algebraic automorphisms by [13]. Thus we need to show now that cases (2)-(6) in Proposition 9.15 indeed present surfaces quasi-homogeneous under the algebraically generated automorphisms. Suppose that $X \rightarrow X_0$ is a minimal resolution of singularities, i.e. X is smooth semi-affine. Consider the most difficult dual graph Γ of $\bar{D} = \bar{X} \setminus X$ as in Figure (5). Making inner blowing up if necessary we can suppose that the weight of C_0 is zero. Let $f : X \rightarrow \mathbb{C}$ be the twisted \mathbb{C}^* -fibration associated with the subgraph $K = \tilde{C}_1 + E + \tilde{C}_2$ and $f_0 : X \rightarrow \mathbb{C}$ be the untwisted \mathbb{C}^* -fibration associated with the 0-vertex C_0 . Suppose that $\bar{f} : \bar{X} \rightarrow \mathbb{P}^1$ and $\bar{f}_0 : \bar{X} \rightarrow \mathbb{P}^1$ are their proper extensions. Let S (resp S_0) be the union of singular fibers of \bar{f} (resp. \bar{f}_0) that meet C_0 (resp. K) and S' (resp. S'_0) be the union of such fibers that do not. That is,

each fiber from S' (resp. S'_0) meets one of curves $C_1, \dots, C_n, E', \tilde{C}'_1, \tilde{C}'_2$. Let U be the open orbit of the natural $\text{AAut}_{\text{hol}}(X)$ -action in X . By Lemma 10.1 $X \setminus U$ is contained in $(S \cup S') \cap (S_0 \cup S'_0)$. On the other hand, since S is not contained in a fiber of \bar{f}_0 , with an exception of a finite set S is contained in U by Lemma 7.5. The same is true for S_0 . Thus $X \setminus U$ is contained in $(S' \cap S'_0) \cup T_0$ where T_0 is a finite set. That is, we need to show that up to a finite set every curve $F \subset S' \cup S'_0$ that is a component of singular fibers of both \bar{f} and \bar{f}_0 is contained in U . Let F meets C_1 . Then by Proposition 2.3 after a sequence of elementary transformations such that all vertices of Γ but C_0 survive we can make the weight of C_1 equal to 0. In particular this 0-vertex yields a \mathbb{C}^* -fibration $f_1 : X \rightarrow \mathbb{C}$. Since F meets C_1 we see that $F \cap X$ is not contained in a fiber of f_1 and therefore by Lemma 7.5 one has $F \setminus T_1 \subset U$ where T_1 is a finite set.

Similarly, by Proposition 2.3 if $i \geq 2$ then after a sequence of elementary transformations under which all vertices $C_i, \dots, C_n, E', \tilde{C}'_1, \tilde{C}'_2$ survive we can make the weight of C_i equal to 0. Thus the same argument implies that if F meets C_i then $F \setminus T_i \subset U$ where T_i is a finite set. In the case of F meeting of the curves E', \tilde{C}'_1 , or \tilde{C}'_2 using elementary transformations in the chain $C_0 + \dots + C_n$ we can make the weight of E' equal to -1 . Then $\tilde{C}'_1 + E' + \tilde{C}'_2$ becomes a subgraph that induces a twisted \mathbb{C}^* -fibration whose restriction to $F \cap X$ is not constant. That is, up to a finite set F is contained in U . Hence we are done in case (5).

The argument for case (3) and a circular graph in case (2) are similar (say, the only difference in (3) is that when one makes the weight of C_n equal to 0 then the associated fibration is a \mathbb{C} -fibration and not a \mathbb{C}^* -fibration).

Also in case (4) the argument is similar, we have to work with the three twisted \mathbb{C}^* -fibrations associated with the three $[-2, -1, -2]$ subgraphs.

If $k' \geq 0$ in case (6) then one needs to make a sequence of inner blow-ups over the edge between E and E' such that the resulting graph looks like

$$\Gamma = \begin{array}{ccccccc} \tilde{C}_1 & & & & & & \tilde{C}'_1 \\ & \swarrow -2 & & & & & \swarrow -2 \\ E & & C_0 & C_1 & \dots & C_{n-1} & C_n & E' \\ & \searrow -2 & \searrow -2 & \searrow -2 & & \searrow -2 & \searrow -1 & \searrow -2 \\ \tilde{C}_2 & & & & & & & \tilde{C}'_2 \end{array}$$

Then we have two twisted \mathbb{C}^* -fibrations $g' : X \rightarrow \mathbb{C}$ and $g : X \rightarrow \mathbb{C}$ induced by the subgraphs $K' = \tilde{C}'_1 + E' + \tilde{C}'_2$ and $K = \Gamma \ominus K'$, indeed K is contractible to a $[-2, -1, -2]$ subgraph by contracting $C_0 + \dots + C_n$. Suppose that \bar{X} is the SNC-completion of X with the boundary described by the graph above and $\bar{g} : \bar{X} \rightarrow \mathbb{P}^1$ (resp. $\bar{g}' : \bar{X} \rightarrow \mathbb{P}^1$) is a proper extension of g (resp. g'). Note that the singular fibers of \bar{g} must meet K' but not K while for the singular fibers of \bar{g}' the situation is reversed. In particular only complete curves that are contained in X may be common components of singular fibers of \bar{g} and \bar{g}' . By Lemma 10.1 U is contained in the complement to the union of such components in X . Hence X_0 is a generalized Gizatullin surface since these components are contractible to points in the Remmert reduction.

In the case of $k' = -1$ the similar argument works and we are done. \square

Propositions 9.15 and 10.2 yield Theorem 1.4 now.

11. HOMOGENEITY

Notation 11.1. In this section X is a smooth affine surface with an SNC-completion \bar{X} such that the dual graph of $\bar{D} = \bar{X} \setminus X$ is one of those in Theorem 1.4. In particular X is a generalized Gizatullin surface.

Note that if X admits a \mathbb{C} or \mathbb{C}^* -fibration without singular fibers (say, X is the complexification of the Klein bottle) then it is homogeneous with respect to $\mathrm{AAut}_{\mathrm{hol}}(X)$ -action because of the absence of fixed points for this action by virtue of Lemma 7.5. The same remains true in several other cases.

Theorem 11.2. *Let \bar{D} have a circular dual graph as in (2) of Theorem 1.4. Then X is homogeneous with respect to $\mathrm{AAut}_{\mathrm{hol}}(X)$ -action.*

Proof. Suppose that C_0, \dots, C_n are irreducible components of \bar{D} such that $C_0^2 = C_1^2 = 0$ and $C_i^1 \leq -1$ for $i \geq 2$, $C_i \cdot C_j = 1$ for $|i - j| = 1$ or $\{i, j\} = \{0, n\}$, and $C_i \cdot C_j = 0$ otherwise. Let $\bar{f} : \bar{X} \rightarrow \mathbb{P}^1$ be the \mathbb{P}^1 -fibration associated with C_0 , the \mathbb{C}^* -fibration f be its restriction to X and let $\{F_i\}$ be the irreducible components of singular fibers of f .

Case 1: $n \geq 3$. The absence of branch points in the dual graph of \bar{D} and the smoothness of X imply that all singular fibers of \bar{f} but the one containing C_2 (say $\bar{f}^{-1}(0)$) are chains $[-1, -1]$ while $\bar{f}^{-1}(0)$ consists of the chain $\mathcal{C} = C_2 + \dots + C_{n-1}$ (joining sections C_1 and C_n) and some other components adjacent to smooth points of this chain which are (-1) -curves because of Lemma 6.3. Each of these (-1) -curves is of course a closure \bar{F}_i of some F_i and it is a component of a singular fiber of \mathbb{P}^1 -fibration associated with C_1 .

Hence by Lemma 10.1 one can suppose that a potential fixed point of the $\mathrm{AAut}_{\mathrm{hol}}(X)$ -action is contained not only in a chain $[-1, -1]$ mentioned before but also in some F_i from $f^{-1}(0)$. Therefore it is enough to show that for any given F_i there is a complete algebraic vector field whose restriction to F_i is locally nilpotent and nontrivial, i.e. it generates a translation on F_i . Though a priori F_i may be adjacent to any C_j with $2 \leq j \leq n$ a reconstruction as in Proposition 2.3 enables us to consider only the case when F_i is adjacent to C_2 .

Contracting in fibers of \bar{f} irreducible component not adjacent to C_1 (in particular C_2 is not contracted) we get a morphism $\varphi : \bar{X} \rightarrow \bar{X}'$ into a Hirzebruch surface \bar{X}' with C'_1 and C'_n playing the roles of disjoint sections where C'_i is the proper transform of C_i in \bar{X}' . That is, $\bar{X}' \setminus C'_0$ is naturally isomorphic to $\mathbb{C}_x^* \times \mathbb{P}_y^1$ with $C'_1 \setminus C'_0, C'_2 \setminus C'_0, C'_n$ given by $\{y = \infty\}, \{x = 0\}, \{y = 0\}$.

Lemma 7.1 implies now that the pull-back of the vector field $\mu = y \frac{\partial}{\partial y}$ on $\bar{X}' \setminus C'_0$ is a rational vector field $\bar{\mu}$ on $\bar{X} \setminus C_0$ which has only simple poles and they are located on those \bar{F}_i 's that are adjacent to the chain \mathcal{C} . This means that $x\mu$ induces a regular vector field ν on X and even on $\bar{X} \setminus C_0$.

Note that $\varphi(F_i) = (0, y_0) \in \mathbb{C}^* \times \mathbb{P}^1$ with $y_0 \neq 0, \infty$ and F_i is obtained as the result of a monoidal transformation at this point. That is, one can introduce a local coordinate system (u, v) on X such that $\varphi(u, v) = (u, uv + y_0)$ and F is given by equation $u = 0$. Then ν is given by $(uv + y_0) \frac{\partial}{\partial v}$, i.e. its restriction to F_i is nonzero and locally nilpotent.

Hence no point of F_i is fixed under the $\text{AAut}_{\text{hol}}(X)$ -action which implies the desired conclusion in this case.

Case 2: $n = 2$. One can blow the edge between C_1 and C_2 to get an extra vertex C , i.e. we have four vertices in the new dual graph. Consider \bar{f} , f , F_i , and $\bar{f}^{-1}(0)$ as before. Note that the weight of the proper transform of C_1 becomes -1 but elementary transformation from Proposition 2.3 can make it again zero while keeping C intact. That is, any F_i from $\bar{f}^{-1}(0)$ is contained in a singular fiber of a \mathbb{C}^* -fibration on X different from f . Lemma 10.1 implies that it suffices again to construct a translation on F_i and the previous argument works.

Case 3: $n = 1$ and \bar{D} consisting of two 0-components C_0 and C_1 meeting each other transversely at two points. It requires a different approach which we sketch below. Let \bar{f} be again the \mathbb{P}^1 -fibration on \bar{X} associated with C_0 . Making contraction $\varphi : \bar{X} \rightarrow \bar{X}'$ in the fibers of \bar{f} we get a Hirzebruch surface \bar{X}' with the proper transform C'_1 of C_1 playing the role of a ramified double section. Let $\bar{g} : \bar{X}' \rightarrow \mathbb{P}_x^1$ be induced by \bar{f} . Without loss of generality we can suppose that $C'_0 = \bar{g}^{-1}(1)$ while 0 and ∞ are the only singular values of $\bar{g}|_{C'_1}$ as in Remark 6.5. Furthermore, applying the same reconstruction we exploited in Proposition 9.11 we can suppose that C'_1 is the closure of the curve given by $x = y^2$ in $\bar{X}' \setminus \bar{g}^{-1}(\infty) \simeq \mathbb{C}_x \times \mathbb{P}_y^1$. That is, the restriction g of \bar{g} to $X' = \bar{X}' \setminus (C'_0 \cup C'_1)$ is a \mathbb{C}^* -fibration with two singular fibers $g^{-1}(0)$ and $g^{-1}(\infty)$ (both isomorphic to \mathbb{C}).

Since X does not contain complete curves the surface \bar{X} is obtained from \bar{X}' by several monoidal transformations at different points of C'_1 . Hence the singular fibers of f are $f^{-1}(0)$, $f^{-1}(\infty)$ (each of them consists of one or two connected components isomorphic to \mathbb{C}) and fibers that are unions of form $F_1 \cup F_2$ where $F_i \simeq \mathbb{C}$ and F_1 meets F_2 transversely at one point.

The vector field $\frac{(x-y^2)}{x-1} \frac{\partial}{\partial y}$ is regular and complete on X' and its restriction to $g^{-1}(0)$ and $g^{-1}(\infty)$ induces nontrivial translations. Furthermore, a calculation shows that it induces a regular vector field ν on X which is a translation on every irreducible component of $f^{-1}(0)$ or $f^{-1}(\infty)$. Therefore, by Lemma 7.1 points of type $F_1 \cap F_2$ are the only potential fixed points of the $\text{AAut}_{\text{hol}}(X)$ -action.

Consider the following reconstruction of the boundary divisor: Blow up one of edges between C_0 and C_1 and contract the proper transform of C_1 . The resulting completion \hat{X} of X has the boundary divisor \hat{D} consisting of two 0-vertices \hat{C}_0 and \hat{C}_1 where \hat{C}_0 is the proper transform of C_0 . Let $\hat{f} : \hat{X} \rightarrow \mathbb{P}^1$ be the \mathbb{P}^1 -fibration on \hat{X} associated with \hat{C}_0 such that $\hat{C}_0 = \hat{f}^{-1}(1)$. Consider the fibers $\hat{F}_1 \cup \hat{F}_2$ of $\hat{f}|_{\hat{X}}$ similar to $F_1 \cup F_2$, i.e. every point that is not of type $\hat{F}_1 \cap \hat{F}_2$ belongs to the open orbit U of the $\text{AAut}_{\text{hol}}(X)$ -action. Note that by construction the proper transform G_i of F_i meets \hat{C}_0 transversely at one point. Hence its intersection with every other fiber of \hat{f} is also 1. In particular, G_i cannot meet the fiber $\hat{F}_1 \cup \hat{F}_2$ at the double point $\hat{F}_1 \cap \hat{F}_2$. This implies that $G_1 \cap G_2 \neq \hat{F}_1 \cap \hat{F}_2$. Thus $F_1 \cap F_2 \in U$ and we are done. \square

Recall that there are Gizatullin surfaces that are not homogeneous with respect to the natural Aut -action. A list of such surfaces appeared in [21] and we show that every surface from this list is homogeneous with respect to the natural AAut_{hol} -action. In

fact this is true for a wider collection of Gizatullin surfaces to describe which we need to remind the following.

Let \bar{Y} be its SNC-completion of a smooth Gizatullin surface Y by a standard zigzag $\bar{D} = \bar{Y} \setminus Y = C_0 \cup \dots \cup C_{n-1}$, $n \geq 3$. The 0-vertices C_0 and C_1 of the zigzag induce two \mathbb{P}^1 -fibrations that lead to a morphism $\bar{\Phi} = (\bar{\varphi}_0, \bar{\varphi}_1) : \bar{Y} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with restriction $\Phi = (\varphi_0, \varphi_1) : Y \rightarrow \mathbb{C}_{x,y}^2$. Omitting a simple case of $n = 3$ we suppose further that $\bar{\Phi}(C_3 \cup \dots \cup C_{n-1}) = (0, 0)$, i.e. the only singular fiber $\bar{\varphi}_0^{-1}(0)$ of $\bar{\varphi}_0$ is contracted by $\bar{\Phi}$ to the proper transform of C_2 . The components of $\bar{\varphi}_0^{-1}(0)$ different from C_2, \dots, C_{n-1} are called feathers (in terminology of [9] or [21]). For every surface in Kovalenko's list each feather is a (-1) -curve.

Theorem 11.3. *Let Y be a smooth Gizatullin surface Y such that every feather is a (-1) -curve. Then Y is homogeneous with respect to the natural AAut_{hol} -action.*

Proof. Since each feather is a (-1) -curve they can be contracted first. This implies that for the sequence $\bar{\Phi} : \bar{Y} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of monoidal transformations, C_{n-1} is obtained from the proper transform $0 \times \mathbb{P}^1$ of C_2 after several (say k) outer blowing-ups in (see Section 2 for definition of outer blowing up) at the origin and infinitely near points. Hence for some fixed values $a_1, \dots, a_{k-1} \in \mathbb{C}$ and general $b \in \mathbb{C}$ the proper transform C of the curve $y = a_1x + \dots + a_{k-1}x^{k-1} + bx^k =: h(x)$ in \bar{Y} meets C_{n-1} at a general point. The triangular automorphism $(x, y) \rightarrow (x, y - h(x))$ of \mathbb{C}^2 induces an isomorphism of Y on another Gizatullin surface Y' which has a completion \bar{Y}' by a standard zigzag $C'_0 + \dots + C'_{n-1}$ such that this isomorphism extends regularly to $\bar{Y} \setminus C_0 \rightarrow \bar{Y}' \setminus C'_0$. We replace \bar{Y} by \bar{Y}' . The advantage is that C is now the proper transform of the x -axis in \mathbb{C}^2 , i.e. it meets both C_{n-1} and C_0 . That is, the graph of $\bar{D} \cup C$ becomes circular with C playing the role of C_n . Thus $X = Y \setminus C$ is a surface of type (2) from Theorem 1.4 and by Theorem 11.2 $\text{AAut}_{\text{hol}}(X)$ acts transitively on X .

Recall that the field ν in Case 1 of Theorem 11.2 extends regularly to $\bar{X} \setminus C_0$ and in particular to $C_n \setminus C_0$ (and therefore to Y). Furthermore, consider transformations $((0, 0, w_3, \dots, w_n)) \rightarrow ((w_3, \dots, w_{j-2}, 0, 0, w_j, \dots, w_n))$ from Proposition 2.3 used in Case 1 to make a feather adjacent to C_2 instead of C_j . Note that C_n survives such a transformation and plays the role of C_{n-j+3} in the modified graph, i.e. it is still contained in $\bar{X} \setminus C_0$. Hence even after these transformations the phase flow of ν is extendable to Y . Since the homogeneity of X is provided by elements of these phase flows we see that X is contained actually in the open orbit of the $\text{AAut}_{\text{hol}}(Y)$ -action. Note that $C \cap Y$ does not contain fixed points of the $\text{AAut}_{\text{hol}}(Y)$ -action since each point of $C \cap Y$ can be moved by a G_a -action induced on Y by the field of form $x^m \frac{\partial}{\partial y}$ on \mathbb{C}^2 with $m \gg 0$. Thus the open orbit coincides with Y . Therefore Y is, indeed, $\text{AAut}_{\text{hol}}(Y)$ -homogeneous. \square

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